

NONCOMMUTATIVE LOCALIZATION AND CHAIN COMPLEXES

I. ALGEBRAIC K - AND L -THEORY

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ABSTRACT. The noncommutative (Cohn) localization $\sigma^{-1}R$ of a ring R is defined for any collection σ of morphisms of f.g. projective left R -modules. We exhibit $\sigma^{-1}R$ as the endomorphism ring of R in an appropriate triangulated category. We use this expression to prove that if $\mathrm{Tor}_i^R(\sigma^{-1}R, \sigma^{-1}R) = 0$ for $i \geq 1$ then every bounded f.g. projective $\sigma^{-1}R$ -module chain complex D with $[D] \in \mathrm{im}(K_0(R) \rightarrow K_0(\sigma^{-1}R))$ is chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective R -module chain complex C , and that there is a localization exact sequence in higher algebraic K -theory

$$\dots \rightarrow K_n(R) \rightarrow K_n(\sigma^{-1}R) \rightarrow K_n(R, \sigma) \rightarrow K_{n-1}(R) \rightarrow \dots,$$

extending to the left the sequence obtained for $n \leq 1$ by Schofield. For a noncommutative localization $\sigma^{-1}R$ of a ring with involution R there are analogous results for algebraic L -theory, extending the results of Vogel from quadratic to symmetric L -theory.

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INTRODUCTION

This is the first of a series of papers on the algebraic K - and L -theory of noncommutative localizations of rings. We adopt throughout the following convention. Suppose R is an associative ring. Unless otherwise specified, by R -module we shall mean left R -module.

Let $\sigma \subset R$ be a multiplicative set of elements in the centre $Z(R)$ of the ring R . It is very classical to define $\sigma^{-1}R$ as the ring of fractions r/s , with $r \in R$ and $s \in \sigma$. The ring $\sigma^{-1}R$ is called the *commutative localization* of R with respect to the multiplicative set σ . Note that the rings R and $\sigma^{-1}R$ are not assumed commutative; the only commutativity hypothesis is that $\sigma \subset Z(R)$. The localization exact sequences relating the algebraic K - and L -groups of R and $\sigma^{-1}R$ are basic computational tools.

More recently, it has turned out to be useful to generalise the notion of rings of quotients, in which much more general σ 's are allowed. From now on, the elements of σ will be maps $s : P \rightarrow Q$, with P and Q f.g. projective R -modules, and localization will invert these maps. The classical case of a multiplicative set $\sigma \subset R$ is just the special case where $P = Q = R$, in other words P and Q are free R modules of rank 1. The morphisms $s : R \rightarrow R$ to be inverted are given by right multiplication by $s \in \sigma$. If all $s \in \sigma$ lie in the centre of R , we are in the traditional situation.

Noncommutative localization is characterized by the following universal property. A ring homomorphism $R \rightarrow S$ is called σ -*inverting* if $1 \otimes s : S \otimes_R P \rightarrow S \otimes_R Q$ is an S -module isomorphism for every $s : P \rightarrow Q$ in σ . The category of σ -inverting ring homomorphisms $R \rightarrow S$ has an initial object, denoted $R \rightarrow \sigma^{-1}R$. This means that any σ -inverting ring homomorphism $R \rightarrow S$ factors uniquely as $R \rightarrow \sigma^{-1}R \rightarrow S$. The ring $\sigma^{-1}R$ is called a *noncommutative localization* or a *universal localization* of R inverting σ .

Noncommutative localization was pioneered by Ore [22] and Cohn [9], in order to study embeddings of noncommutative rings in skewfields. See Ranicki [25] for some of the applications of the algebraic K - and L -theory of noncommutative localization to the topology of codimension 2 submanifolds, such as knots.

In Part I of the paper we study the algebraic K - and L -theory of a noncommutative localization $\sigma^{-1}R$ by means of triangulated categories, generalizing the work of Vogel [32] and Schofield [27]. In Part II we shall obtain a chain complex interpretation of the normal form of Gerasimov [13] and Malcolmson [19] for elements of $\sigma^{-1}R$.

The ring homomorphism $R \rightarrow \sigma^{-1}R$ gives $\sigma^{-1}R$ the structure of a right R -module in the usual manner, and we have a functor

$$\begin{aligned} \sigma^{-1} &= \sigma^{-1}R \otimes_R - : \{R\text{-modules}\} \rightarrow \{\sigma^{-1}R\text{-modules}\} ; \\ M &\mapsto \sigma^{-1}M = \{\sigma^{-1}R\} \otimes_R M . \end{aligned}$$

A $\sigma^{-1}R$ -module is *induced f.g. projective* if it is of the form $\sigma^{-1}P$ for a f.g. projective R -module P .

The *chain complex lifting problem* is to decide if a bounded chain complex D of induced f.g. projective $\sigma^{-1}R$ -modules is chain equivalent to $\sigma^{-1}C$ for a bounded chain complex C of f.g. projective R -modules. The problem has a trivial affirmative solution for a commutative localization, by the clearing of denominators, when D is actually isomorphic to $\sigma^{-1}C$. In Part I of the paper we apply triangulated categories to study the problem for a noncommutative localization $\sigma^{-1}R$.

A systematic solution of the chain complex lifting problem leads to the extensions to the noncommutative case of the localization exact sequences for the algebraic K - and L -groups of a commutative localization. It turns out that the problem has a systematic solution if and only if $\sigma^{-1}R$ is ‘stably flat over R ’, and that there are homological obstructions to stable flatness in general. Here is what we mean by stably flat over R .

Definition 0.1. Let $R \longrightarrow S$ be a ring homomorphism. The ring S is called *stably flat over R* if :

- (i) the multiplication map $\mu : S \otimes_R S \longrightarrow S$ is an isomorphism,
- (ii) $\mathrm{Tor}_i^R(S, S) = 0$ for all $i \geq 1$.

□

Remark 0.2. In the case of a noncommutative localization $R \longrightarrow \sigma^{-1}R$, it is always true that $\mu : S \otimes_R S \longrightarrow S$ is an isomorphism, and that $\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0$. A proof may be found on page 58 of Schofield [27], or also in Corollary 3.27 of this article. In general $\mathrm{Tor}_i^R(\sigma^{-1}R, \sigma^{-1}R) \neq 0$ for $i \geq 2$.

□

In fact, Schofield has constructed examples of noncommutative localizations $\sigma^{-1}R$ which are not stably flat over R , with $\mathrm{Tor}_2^R(\sigma^{-1}R, \sigma^{-1}R) \neq 0$. These examples pinpoint the difference between commutative and noncommutative localization. In commutative localization $\sigma^{-1}R$ is always a flat R -module (Example 0.3 below); Schofield’s examples show that in noncommutative localization the ring $\sigma^{-1}R$ need not even be stably flat over R . We shall describe these examples in Part II.

Example 0.3. Let $\sigma \subset R$ be a multiplicatively closed subset with $1 \in \sigma$, such that the elements $s \in \sigma$ are central in R or more generally satisfy the *Ore conditions* :

- (i) for all $r \in R$, $s \in \sigma$ there exist $q \in R$, $t \in \sigma$ such that $rt = sq \in R$,
- (ii) if $r \in R$, $s \in \sigma$ are such that $sr = 0 \in R$ then $rt = 0 \in R$ for some $t \in \sigma$.

The *Ore localization* $\sigma^{-1}R$ is the ring of fractions r/s ($r \in R, s \in \sigma$), which are the equivalence classes of pairs (r, s) with

$$(r, s) \sim (q, t) \text{ if there exist } u, v \in R \text{ such that } ru = qv \in R, su = tv \in \sigma.$$

An Ore localization $\sigma^{-1}R$ is flat (Stenström [28], Prop. II.3.5) and hence stably flat over R .

□

Example 0.4. Recall that a ring R is called *hereditary* if all R -modules have projective dimension ≤ 1 . A noncommutative localization of a hereditary ring R is a hereditary ring $\sigma^{-1}R$ (Bergman and Dicks [4]). In this case, the vanishing of $\text{Tor}_i^R(\sigma^{-1}R, \sigma^{-1}R)$ with $i \geq 2$ follows just from the fact that every R -module has projective dimension ≤ 1 . Thus $\sigma^{-1}R$ is stably flat over R .

□

Example 0.5. For $\mu \geq 1$ let F_μ be the free group on μ generators. Given a commutative ring k let $R = k[F_\mu]$, and let σ be the set of all R -module morphisms $s : R^n \rightarrow R^n$ ($n \geq 1$) inducing k -module isomorphisms $\epsilon(s) : k^n \rightarrow k^n$ via the augmentation $\epsilon : R \rightarrow k$. The noncommutative localization $\sigma^{-1}R$ is flat for $\mu = 1$ (when it is commutative), but is not flat for $\mu \geq 2$. If k is a principal ideal domain Farber and Vogel [11] identified $\sigma^{-1}R$ with the ring of rational functions in μ non-commuting variables, and proved that $\sigma^{-1}R$ is stably flat over R . (If k is a field then $k[F_\mu]$ is hereditary by Cohn [8], and the stable flatness is given by Example 0.4).

□

Now that we have seen some examples of stably flat localizations $R \rightarrow \sigma^{-1}R$, it is time for a comment. The main results of the article are about stably flat localizations. We do have some weak results that hold without the hypothesis of stable flatness; but the powerful theorems assume that $\sigma^{-1}R$ is stably flat over R . It becomes interesting to find equivalent formulations of the hypothesis that $\sigma^{-1}R$ is stably flat over R .

Equivalent Formulation 0.6. Let $R \rightarrow S$ be a ring homomorphism. The ring S is stably flat over R if and only if

- (i) the multiplication map $\mu : S \otimes_R S \rightarrow S$ is an isomorphism,
- (ii) for all S -modules M and all $i \geq 1$, we have $\text{Tor}_i^R(S, M) = 0$.

The if implication is trivial; if $\text{Tor}_i^R(S, M)$ vanishes for all S -modules M , then it vanishes in particular for $M = S$. The only if implication may be found in Lemma 3.30.

□

Remark 0.7. (i) Equivalent Formulation 0.6 should explain the terminology. S is flat as a right R -module if and only if $\text{Tor}_i^R(S, M)$ vanishes for all R -modules M , while S is stably flat over R if and only if $\text{Tor}_i^R(S, M)$ vanishes for all S -modules M . In 0.13 we shall see an equivalent formulation of stable flatness, this time in terms of Vogel's chain complex $E(C)$. It seems natural to postpone this equivalent formulation until we are ready to discuss Vogel's construction.

(ii) A ring homomorphism $R \longrightarrow S$ such that $\mathrm{Tor}_i^R(S, M) = 0$ for all S -modules M and all $i \geq 1$ is called a *lifting* by Dicks [10] (p. 565).

□

Let \mathcal{C} be a triangulated category. We say that \mathcal{C} satisfies [TR5] if arbitrary coproducts exist in \mathcal{C} . An object c in a triangulated category \mathcal{C} satisfying [TR5] is called *compact* if

$$\mathcal{C}\left(c, \coprod_{\lambda \in \Lambda} t_\lambda\right) = \bigoplus_{\lambda \in \Lambda} \mathcal{C}(c, t_\lambda)$$

for every collection $\{t_\lambda \mid \lambda \in \Lambda\}$ of objects in \mathcal{C} . Let $\mathcal{C}^c \subset \mathcal{C}$ be the full subcategory of all the compact objects in \mathcal{C} . For the derived category $D(R)$ of unbounded R -module chain complexes, the compact category $D(R)^c = D^c(R) \subset D(R)$ has for its objects the bounded f.g. projective R -module chain complexes, and any objects isomorphic to these. Let $D(R, \sigma) \subset D(R)$, $D^c(R, \sigma) \subset D^c(R)$ be the subcategories generated by σ . The objects of $D^c(R, \sigma)$ are the bounded f.g. projective R -module chain complexes C such that $H_*(\sigma^{-1}C) = 0$ (Proposition 5.3). We shall be working with triangulated categories and functors

$$\begin{array}{ccccccc} \mathcal{A}^c = D^c(R, \sigma) & \longrightarrow & \mathcal{B}^c = D^c(R) & \xrightarrow{\pi} & \mathcal{C}^c & \xrightarrow{T} & \mathcal{D}^c = D^c(\sigma^{-1}R) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} = D(R, \sigma) & \longrightarrow & \mathcal{B} = D(R) & \xrightarrow{\pi} & \mathcal{C} = D(R)/D(R, \sigma) & \xrightarrow{T} & \mathcal{D} = D(\sigma^{-1}R) \end{array}$$

The unnamed maps are inclusions. The natural map $\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c$ is an idempotent completion (see 3.9.4). The functor $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ is the projection to the quotient. The functor

$$T\pi : \mathcal{B} \longrightarrow \mathcal{D} ; X \mapsto \{\sigma^{-1}R\}^L \otimes_R X = \sigma^{-1}P$$

is given by the tensor ${}^L\otimes_R$ in the derived category, constructed using a sufficiently nice projective R -module chain complex P with a homology equivalence $P \longrightarrow X$. And the functor T is determined as the unique factorization of $T\pi$ through π . We shall be particularly concerned with the extent to which $T = \sigma^{-1} : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ is an equivalence of triangulated categories.

Here is our main result (Theorem 10.8) :

Theorem 0.8. *The following conditions on a noncommutative localization $R \longrightarrow \sigma^{-1}R$ are equivalent :*

- (i) $\sigma^{-1}R$ is stably flat over R ,
- (ii) the functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ is an equivalence of categories.

□

Our main Theorem 0.8 has an immediate consequence (Theorem 10.10) :

Theorem 0.9. *If $\sigma^{-1}R$ is stably flat over R then the chain complex lifting problem can always be solved : for every bounded chain complex D of induced f.g. projective $\sigma^{-1}R$ -modules there exists a bounded f.g. projective R -module chain complex C with a chain equivalence $\sigma^{-1}C \simeq D$. \square*

Without the stable flatness hypothesis there are Toda bracket obstructions to lifting chain complexes; we explain this briefly in Section 6. A much fuller treatment will come in Part II of this series.

Let $H(R, \sigma)$ be the exact category of σ -torsion R -modules T of projective dimension 1, i.e. the R -modules with a f.g. projective R -module resolution

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

such that $\sigma^{-1}s : \sigma^{-1}P \longrightarrow \sigma^{-1}Q$ is an isomorphism (e.g. if $s \in \sigma$). We shall only be dealing with $H(R, \sigma)$ in the special case of a noncommutative localization $\sigma^{-1}R$ when each morphism $s : P \longrightarrow Q$ in σ is injective. This happens, for example, if $R \longrightarrow \sigma^{-1}R$ is injective (see Proposition 11.2 for details).

The algebraic K -theory localization exact sequence for an injective Ore localization $R \longrightarrow \sigma^{-1}R$

$$\dots \longrightarrow K_n(R) \longrightarrow K_n(\sigma^{-1}R) \longrightarrow K_n(R, \sigma) \longrightarrow K_{n-1}(R) \longrightarrow \dots$$

was obtained by Bass [2] for $n = 1$ and Gersten [14], Quillen [23] and Grayson [15], [16] for $n \geq 2$, with $K_*(R, \sigma) = K_{*-1}(H(R, \sigma))$. Schofield [27] established the algebraic K -theory localization exact sequence in the classical dimensions 0, 1

$$K_1(R) \longrightarrow K_1(\sigma^{-1}R) \longrightarrow K_1(R, \sigma) \longrightarrow K_0(R) \longrightarrow K_0(\sigma^{-1}R)$$

for any injective noncommutative localization $R \longrightarrow \sigma^{-1}R$. It is easy to extend the exact sequence to the right, using the lower K -groups K_{-*} of [2].

Waldhausen [33] identified the algebraic K -groups of $D^c(R)$ with the Quillen [23] algebraic K -groups of R

$$K_*(D^c(R)) = K_*(R) .$$

By the localization theorem of [33], for any R, σ there is defined a long exact sequence in algebraic K -theory

$$\dots \longrightarrow K_n(R) \longrightarrow K_n(D^c(R)/D^c(R, \sigma)) \longrightarrow K_n(R, \sigma) \longrightarrow K_{n-1}(R) \longrightarrow \dots .$$

with

$$K_*(R, \sigma) = K_{*-1}(D^c(R, \sigma)) \text{ (definition) .}$$

The idempotent completion $D^c(R)/D^c(R, \sigma) \longrightarrow \mathcal{C}^c$ induces an isomorphism on higher K -theory, so that

$$K_*(D^c(R)/D^c(R, \sigma)) = K_*(\mathcal{C}^c) .$$

The map $T : \mathcal{C}^c \longrightarrow D^c(\sigma^{-1}R)$ induces a map on Waldhausen's K -theory

$$K(T) : K(\mathcal{C}^c) \longrightarrow K(D^c(\sigma^{-1}R)) = K(\sigma^{-1}R) .$$

In Sections 8 and 9 we show that this map is an isomorphism on K_0 and K_1 , without any hypothesis on σ . If we assume that $\sigma^{-1}R$ is stably flat over R , then Theorem 10.8 tells us that $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$ is an equivalence of categories, and as a corollary we deduce Theorem 10.11 :

Theorem 0.10. *If $\sigma^{-1}R$ is stably flat over R the functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$ induces isomorphisms*

$$T : K_*(\mathcal{C}^c) = K_*(D^c(R)/D^c(R, \sigma)) \longrightarrow K_*(\mathcal{D}^c) = K_*(\sigma^{-1}R)$$

and there is a localization exact sequence in algebraic K -theory

$$\dots \longrightarrow K_n(R) \longrightarrow K_n(\sigma^{-1}R) \longrightarrow K_n(R, \sigma) \longrightarrow K_{n-1}(R) \longrightarrow \dots \quad \square$$

In Theorem 11.10 we shall prove :

Theorem 0.11. *If each morphism in σ is injective the Waldhausen K -groups of $D^c(R, \sigma)$ are just the Quillen K -groups of $H(R, \sigma)$*

$$K_*(R, \sigma) = K_{*-1}(D^c(R, \sigma)) = K_{*-1}(H(R, \sigma)) .$$

If in addition $\sigma^{-1}R$ is stably flat over R there is defined a localization exact sequence in algebraic K -theory

$$\dots \longrightarrow K_n(R) \longrightarrow K_n(\sigma^{-1}R) \longrightarrow K_{n-1}(H(R, \sigma)) \longrightarrow K_{n-1}(R) \longrightarrow \dots \quad \square$$

However, in general the morphisms $s : P \longrightarrow Q$, $R \longrightarrow \sigma^{-1}R$ are not injective and, except in section 11 and parts of section 12, we do not assume this to be the case.

In section 12 we consider the L -theory of noncommutative localizations, obtaining the following results (Theorems 12.4, 12.5, 12.9) :

Theorem 0.12. *Let $R \longrightarrow \sigma^{-1}R$ be a noncommutative localization of a ring with involution R , such that σ is invariant under the involution.*

(i) *There is a localization exact sequence of quadratic L -groups*

$$\dots \longrightarrow L_n(R) \longrightarrow L_n^I(\sigma^{-1}R) \xrightarrow{\partial} L_n(R, \sigma) \longrightarrow L_{n-1}(R) \longrightarrow \dots$$

with $I = \text{im}(K_0(R) \longrightarrow K_0(\sigma^{-1}R))$, and $L_n(R, \sigma)$ the cobordism group of $\sigma^{-1}R$ -contractible $(n-1)$ -dimensional quadratic Poincaré complexes over R .

(ii) *If $\sigma^{-1}R$ is stably flat over R there is a localization exact sequence of symmetric L -groups*

$$\dots \longrightarrow L^n(R) \longrightarrow L_I^n(\sigma^{-1}R) \xrightarrow{\partial} L^n(R, \sigma) \longrightarrow L^{n-1}(R) \longrightarrow \dots$$

with $L^n(R, \sigma)$ the cobordism group of $\sigma^{-1}R$ -contractible $(n-1)$ -dimensional symmetric Poincaré complexes over R .

(iii) If $R \longrightarrow \sigma^{-1}R$ is injective then $L^n(R, \sigma)$ (resp. $L_n(R, \sigma)$) is the cobordism group of n -dimensional symmetric (resp. quadratic) Poincaré complexes of σ -torsion R -modules of projective dimension 1. \square

The L -theory exact sequences of Theorem 0.12 for an injective Ore localization $R \longrightarrow \sigma^{-1}R$ were obtained in Ranicki [24]. The quadratic L -theory exact sequence of 0.12 (i) for arbitrary $R \longrightarrow \sigma^{-1}R$ was obtained by Vogel [31], [32]. The symmetric L -theory exact sequence of 0.12 (ii) is new.

We shall now give yet another equivalent formulation of stable flatness. However, in Part I, we will use Definition 0.1 as the working definition.

Recall that a right R -module S is flat if and only if, for any chain complex C of left R -modules

$$H_*(S^L \otimes_R C) = S \otimes_R H_*(C)$$

where S^L is a projective right R -module resolution of S .

Equivalent Formulation 0.13. For any noncommutative localization $\sigma^{-1}R$ and any chain complex of R -modules C , we define a contravariant functor

$$\begin{aligned} [[-, C]] : \{R\text{-module chain complexes}\} &\longrightarrow \{\mathbb{Z}\text{-modules}\} ; \\ A \mapsto [[A, C]] &= \varinjlim_{(B, \beta)} [A, B] \end{aligned}$$

with $[A, B]$ the \mathbb{Z} -module of chain homotopy classes of chain maps $A \longrightarrow B$, and the direct limit taken over all the R -module chain complexes B with a chain map $\beta : C \longrightarrow B$ such that the mapping cone of β is quasi-isomorphic to a bounded complex of f.g. projective R -modules, and $H_*(S^L \otimes_R C) \cong H_*(S^L \otimes_R B)$ with $S = \sigma^{-1}R$. (A chain map is a quasi-isomorphism if it induces isomorphisms in homology). This functor is representable. That is

$$[[A, C]] = [A, E(C)]$$

for some R -module chain complex $E(C) = \varinjlim B$. Such a complex $E(C)$ was first constructed by Vogel in his paper [32]. There is a map of R -module chain complexes $C \longrightarrow E(C)$ inducing S -module isomorphisms

$$H_*(S^L \otimes_R C) \cong H_*(S^L \otimes_R E(C))$$

and such that for each $i \in \mathbb{Z}$

$$H_i(E(C)) = [[\Sigma^i R, C]] = \varinjlim_{(B, \beta)} H_i(B)$$

is an S -module. Furthermore, $H_0(E(R)) = S$. If $H_i(C)$ are S -modules for all $i \in \mathbb{Z}$, then the map $C \longrightarrow E(C)$ is a quasi-isomorphism. There is a map of R -module chain complexes $E(C) \longrightarrow S^L \otimes_R C$ inducing S -module morphisms

$$H_*(E(C)) \longrightarrow H_*(S^L \otimes_R C) \cong H_*(S^L \otimes_R E(C)) .$$

Here is the equivalent formulation : the noncommutative localization $S = \sigma^{-1}R$ is stably flat over R if and only if $E(R) \longrightarrow S^L \otimes_R R = S$ is a quasi-isomorphism. The proof of this assertion is the equivalence of (i) and (ii) in Theorem 10.8, coupled with the fact that $E(R) = G\pi R$. If S is stably flat then $E(C) \longrightarrow S^L \otimes_R C$ is a quasi-isomorphism for any R -module chain complex C . The proof of this statement comes from the fact that the full subcategory on which the map is an isomorphism is triangulated and closed under coproducts, coupled with Lemma 3.5. In Part II of this series we plan to explore in far greater depth the relation between our work and the earlier work in the subject, especially the important contributions of Vogel.

□

So far, we have used chain complexes and homology. This makes the notation above consistent with the the usage standard in the L -theory literature. For triangulated categories, it is more usual to work with cochain complexes and cohomology, and in Part I we shall be working with these (except in the L -theory section 12). The sole object is to make it easier for the reader to check our references to the literature.

The basic tool in the proof of our main Theorem 0.8 is the fact that

$$\pi : \mathcal{B} = D(R) \longrightarrow \mathcal{C} = D(R)/D(R, \sigma)$$

admits a Bousfield localization. That is, π has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$, meaning

$$\mathcal{B}(x, Gy) = \mathcal{C}(\pi x, y)$$

for any objects $x \in \mathcal{B}$, $y \in \mathcal{C}$. The R -module cochain complex $G\pi R$ (which is essentially the same as Vogel's $E(R)$; see Equivalent Formulation 0.13 above) has the following properties :

- (i) $H^{-i}(G\pi R)$ is the group of morphisms $\Sigma^i R \longrightarrow R$ in \mathcal{C} . Such morphisms are equivalence classes of pairs of R -module chain maps $(\alpha : R \longrightarrow Y, \beta : \Sigma^i R \longrightarrow Y)$ with $Y \in D^c(R)$, $C(\alpha) \in D^c(R, \sigma)$. The fact that Y may be taken to be in $D^c(R) \subset D(R)$ is not supposed to be trivial: see 3.9.4.
- (ii) The groups $H^*(G\pi R)$, $\mathrm{Tor}_*^R(\sigma^{-1}R, \sigma^{-1}R)$ are $\sigma^{-1}R$ - $\sigma^{-1}R$ bimodules, and there is a cohomology spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^R(\sigma^{-1}R, H^j(G\pi R)) \implies H^*(\{\sigma^{-1}R\}^L \otimes_R G\pi R) = \sigma^{-1}R,$$

with $\sigma^{-1}R$ concentrated in degree 0. In particular

$$H^0(G\pi R) = \sigma^{-1}R.$$

- (iii) $H^i(G\pi R) = 0$ for $i \neq 0$ if and only if $\sigma^{-1}R$ is stably flat over R .

1. NONCOMMUTATIVE LOCALIZATION OF A RING

Given a ring R and a collection σ of morphisms $s : P \longrightarrow Q$ of f.g. projective R -modules we recall the universal property of the noncommutative localization $\sigma^{-1}R$, and the original construction.

Definition 1.1. (i) A ring homomorphism $R \longrightarrow S$ is σ -*inverting* if for every $s : P \longrightarrow Q$ in σ the induced S -module morphism $1 \otimes s : S \otimes_R P \longrightarrow S \otimes_R Q$ is an isomorphism. (ii) A ring homomorphism $R \longrightarrow S$ is *universally σ -inverting* if it is σ -inverting, and any other σ -inverting morphism $R \longrightarrow S'$ has a unique factorization $R \longrightarrow S \longrightarrow S'$.

□

Any two universally σ -inverting ring homomorphisms $R \longrightarrow S, R \longrightarrow S'$ are related by a canonical isomorphism $S \longrightarrow S'$ such that $R \longrightarrow S'$ is the composite $R \longrightarrow S \longrightarrow S'$.

Theorem 1.2. (Cohn [9])

For any R, σ there exists a universally σ -inverting ring morphism $R \longrightarrow \sigma^{-1}R$.

□

As in the introduction, for any R -module M we define $\sigma^{-1}M = \{\sigma^{-1}R\} \otimes_R M$. Because $R \longrightarrow \sigma^{-1}R$ is σ -inverting, every $s : P \longrightarrow Q$ in σ induces an isomorphism $s : \sigma^{-1}P \longrightarrow \sigma^{-1}Q$.

The original construction of $\sigma^{-1}R$ in [9] was for a set σ of R -module morphisms $s : R^n \longrightarrow R^n$, i.e. for a set of square matrices s , with $\sigma^{-1}R$ obtained from R by adjoining one generator for each component of a formal inverse s^{-1} and the relations given by

$$ss^{-1} = s^{-1}s = I .$$

Gerasimov [13], Malcolmson [19] and Schofield [27] constructed $\sigma^{-1}R$ for any σ as the ring of equivalence classes of triples of morphisms of f.g. projective R -module morphisms

$$(f : P \longrightarrow R, s : P \longrightarrow Q, g : R \longrightarrow Q)$$

with $s \in \sigma$. The triple (f, s, g) represents $fs^{-1}g \in \sigma^{-1}R$.

2. BOUSFIELD LOCALIZATION IN TRIANGULATED CATEGORIES

In section 3 we shall express a noncommutative localization $\sigma^{-1}R$ of a ring R as the endomorphism of R in the triangulated category $\mathcal{C}^c = (D(R)/D(R, \sigma))^c$

$$\sigma^{-1}R = \text{End}_{\mathcal{C}^c}(R) .$$

The main tool is Bousfield localization, which we review in this section. We give careful statements of what is known, and refer elsewhere for the proofs.

Definition 2.1. Let \mathcal{B} be a triangulated category. A *triangulated subcategory* is a non-empty full subcategory $\mathcal{A} \subset \mathcal{B}$ closed under suspension and triangles; that is, given a distinguished triangle in \mathcal{B}

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X ,$$

if X and Y lie in \mathcal{A} then so do all their suspensions, and so does Z .

□

Definition 2.2. Let \mathcal{B} be a triangulated category. A triangulated subcategory $\mathcal{A} \subset \mathcal{B}$ is called *thick* (or *épaisse*) if it contains all direct summands of its objects.

□

And now we get to the first theorem :

Theorem 2.3. (Verdier localization). *Let \mathcal{B} be a triangulated category, $\mathcal{A} \subset \mathcal{B}$ a triangulated subcategory. There is a quotient triangulated category $\mathcal{C} = \mathcal{B}/\mathcal{A}$, and a natural triangulated functor $\mathcal{B} \longrightarrow \mathcal{C}$. The composite functor*

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$$

takes every object in \mathcal{A} to an object isomorphic in \mathcal{C} to zero. The functor $\mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A}$ is universal among the functors $\mathcal{B} \longrightarrow \mathcal{D}$ taking the objects of \mathcal{A} to objects isomorphic to zero. Furthermore, if $\mathcal{A} \subset \mathcal{B}$ is thick, then all the objects in \mathcal{B} whose images in $\mathcal{C} = \mathcal{B}/\mathcal{A}$ are isomorphic to zero lie in \mathcal{A} .

Proof. The theorem is due to Verdier, and may be found in his thesis [30]. For a very complete and detailed proof, see Theorem 2.1.8, on page 74 of [20]. The full proof occupies pages 75-99 op. cit. □

Suppose $\mathcal{C} = \mathcal{B}/\mathcal{A}$ is as in Theorem 2.3. It may happen that the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C}$ has a right adjoint, that is a functor $G : \mathcal{C} \longrightarrow \mathcal{B}$ such that for every object x in \mathcal{B} and every object y in \mathcal{C}

$$\mathcal{B}(x, Gy) = \mathcal{C}(\pi x, y) .$$

We shall be working in the following situation :

Definition 2.4. Let \mathcal{B} be a triangulated category, and let $\mathcal{A} \subset \mathcal{B}$ be a thick subcategory. Let $\mathcal{C} = \mathcal{B}/\mathcal{A}$ be the quotient of Theorem 2.3. We say that a *Bousfield localization functor exists for the pair $\mathcal{A} \subset \mathcal{B}$* if the natural functor $\pi : \mathcal{B} \longrightarrow \mathcal{C}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$.

□

Remark 2.5. The adjoint G , if it exists, must be a triangulated functor. Adjoints of triangulated functors are always triangulated. See Lemma 5.3.6 of [20].

□

Let us next summarise some useful facts about Bousfield localizations. This is not an exhaustive list; later on in the article we shall cite more properties. What comes here is a handy list of basic, core properties.

Theorem 2.6. *Let \mathcal{B} be a triangulated category, and let $\mathcal{A} \subset \mathcal{B}$ be a thick subcategory. Suppose a Bousfield localization functor exists for the pair $\mathcal{A} \subset \mathcal{B}$. That is, the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$. Then the following statements are true.*

2.6.1. *Coproducts in \mathcal{B} of objects of \mathcal{A} must lie in \mathcal{A} .*

2.6.2. *The functor G is fully faithful.*

2.6.3. *For any two objects x and y in \mathcal{B} , we have an isomorphism*

$$\mathcal{B}(G\pi x, G\pi y) \longrightarrow \mathcal{B}(x, G\pi y) ; \alpha \mapsto \alpha\eta_x ,$$

where η is the unit of adjunction, and $\alpha\eta_x$ is the composite

$$\alpha\eta_x : x \xrightarrow{\eta_x} G\pi x \xrightarrow{\alpha} G\pi y .$$

The assertion is really that any $\beta : x \longrightarrow G\pi y$ factors uniquely as $\alpha\eta_x$.

2.6.4. *If $b \in \mathcal{B}$ lies in the image of $G : \mathcal{C} \longrightarrow \mathcal{B}$, then $\eta_t : b \longrightarrow G\pi b$ is an isomorphism.*

2.6.5. *Let $b \in \mathcal{B}$ be any object. The unit of adjunction gives a map $\eta_b : b \longrightarrow G\pi b$. Complete it to a triangle*

$$a \longrightarrow b \xrightarrow{\eta_b} G\pi b \longrightarrow \Sigma a .$$

Then the object a lies in $\mathcal{A} \subset \mathcal{B}$.

Proof. To prove 2.6.1, just observe that the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C}$ has a right adjoint, and therefore takes coproducts to coproducts. Now π takes objects of \mathcal{A} to zero, and hence takes any coproduct of objects in \mathcal{A} to zero. But because \mathcal{A} is thick, Theorem 2.3 tells us that any object of \mathcal{B} whose image in \mathcal{C} vanishes must lie in \mathcal{A} . Hence any \mathcal{B} -coproduct of objects of \mathcal{A} lies in \mathcal{A} .

Now for 2.6.2. In the proof of Lemma 9.1.7 of [20] we see that, for all $x \in \mathcal{B}$, the map $\varepsilon_{\pi x}$ is an isomorphism. Any object of \mathcal{C} is of the form πx , hence $\varepsilon : \pi G \Longrightarrow 1$ is an isomorphism. But then Lemma A.2.9 of [20] tells us that G is fully faithful.

Next comes 2.6.3. We have

$$\begin{aligned} \mathcal{B}(x, G\pi y) &= \mathcal{C}(\pi x, \pi y) && \text{by adjunction,} \\ &= \mathcal{B}(G\pi x, G\pi y) && \text{because } G \text{ is fully faithful.} \end{aligned}$$

Finally, 2.6.4 may be found in Lemma 9.1.7 of [20], while 2.6.5 may be found in Proposition 9.1.8 loc. cit. \square

It might be useful to illustrate these properties in the case of most interest to us, with $\mathcal{B} = D(R)$ the (unbounded) derived category of chain complexes of left R -modules. When we refer to the object $R \in D(R)$, we mean the chain complex which is R in degree 0, and zero elsewhere. Let us make an observation.

Proposition 2.7. *Let $\mathcal{B} = D(R)$ be the derived category of a ring R . Let $\mathcal{A} \subset \mathcal{B} = D(R)$ be a thick subcategory. Suppose a Bousfield localization functor exists for the pair $\mathcal{A} \subset D(R)$. That is, the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$. Then $G\pi R$ is a chain complex in $D(R)$. We assert that $S = H^0(G\pi R)$ has a natural structure of an algebra over R , i.e. there exists a ring homomorphism $R \longrightarrow S$.*

Proof. We note first that $H^0(X) = \mathcal{B}(R, X)$ for every object $X \in \mathcal{B} = D(R)$. Applying this to $X = G\pi R$, we have

$$\begin{aligned} H^0(G\pi R) &= \mathcal{B}(R, G\pi R) && \text{by the above,} \\ &= \mathcal{B}(G\pi R, G\pi R) && \text{by 2.6.3.} \end{aligned}$$

Now $S = \mathcal{B}(G\pi R, G\pi R)$ is a ring, being the endomorphism ring of an object in an additive category. The fact that $G\pi$ is an additive functor gives us a ring homomorphism

$$\mathcal{B}(R, R) \longrightarrow \mathcal{B}(G\pi R, G\pi R) = S .$$

Let us agree that we shall view R and $G\pi R$ as right modules for, respectively, $\mathcal{B}(R, R)$ and $\mathcal{B}(G\pi R, G\pi R)$. Then $\mathcal{B}(R, R) = R$ with the usual right action, and we have a homomorphism of rings

$$R \longrightarrow \mathcal{B}(G\pi R, G\pi R) = S .$$

□

Lemma 2.8. *Let the situation be as in Proposition 2.7. We remind the reader: $\mathcal{B} = D(R)$ is the derived category of a ring R . Let $\mathcal{A} \subset \mathcal{B} = D(R)$ be a thick subcategory. Suppose a Bousfield localization functor exists for the pair $\mathcal{A} \subset D(R)$. That is, the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$. The unit of adjunction $\eta_R : R \longrightarrow G\pi R$ induces a map*

$$H^0(\eta_R) : R = H^0(R) \longrightarrow H^0(G\pi R) .$$

We assert that it agrees with the ring homomorphism of Proposition 2.7.

Proof. Recall that R can be viewed as $\text{Hom}_R(R, R)$, acting on the right. For any $r \in R$, right multiplication by r is a left-module homomorphism. We denote this homomorphism as $r : R \longrightarrow R$.

The naturality of η gives a commutative square

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & G\pi R \\ \downarrow r & & \downarrow G\pi r \\ R & \xrightarrow{\eta_R} & G\pi R \end{array}$$

Taking H^0 , we have a commutative square

$$\begin{array}{ccc} R & \xrightarrow{H^0(\eta_R)} & H^0(G\pi R) \\ \downarrow r & & \downarrow G\pi r \\ R & \xrightarrow{H^0(\eta_R)} & H^0(G\pi R) \end{array}$$

Since this commutes for any $r \in R$, we deduce that $H^0(\eta_R) : R \longrightarrow H^0(G\pi R)$ is a homomorphism of right R -modules. Here, R is a right R -modules in the obvious way, while $S = H^0(G\pi R)$ is a right module over the ring $S = \mathcal{B}(G\pi R, G\pi R)$, and the ring homomorphism $R \longrightarrow S$ of Proposition 2.7 turns it into a right R -module. To prove that the R -module homomorphism $H^0(\eta_R) : R \longrightarrow H^0(G\pi R)$ agrees with the ring homomorphism $R \longrightarrow S$, we need only check that $1 \in R$ maps to $1 \in S$.

But this is easy. For any element $s \in S$, the identifications

$$s \in S = H^0(G\pi R) = \mathcal{B}(R, G\pi R) = \mathcal{B}(G\pi R, G\pi R)$$

are quite explicit. Given $\alpha \in \mathcal{B}(G\pi R, G\pi R)$, that is a morphism $\alpha : G\pi R \longrightarrow G\pi R$, the isomorphism

$$\mathcal{B}(R, G\pi R) = \mathcal{B}(G\pi R, G\pi R)$$

of 2.6.3 sends it to the composite

$$R \xrightarrow{\eta_R} G\pi R \xrightarrow{\alpha} G\pi R .$$

In particular, if $\alpha = 1_S$, then it is sent to η_R . The isomorphism

$$H^0(G\pi R) = \mathcal{B}(R, G\pi R)$$

sends $\eta_R \in \mathcal{B}(R, G\pi R)$ to the image of $1_R \in R = H^0(R)$ under the map

$$H^0(\eta_R) : R \longrightarrow H^0(G\pi R) .$$

We conclude that under the natural isomorphisms, $1_S = H^0(\eta_R)(1_R)$. This proves that $H^0(\eta_R)$ takes $1 \in R$ to $1 \in S$. \square

3. NONCOMMUTATIVE LOCALIZATION USING TRIANGULATED CATEGORIES

In Lemma 2.8 we learned that, given a suitable subcategory $\mathcal{A} \subset \mathcal{B} = D(R)$ for which the projection $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$, there is a canonical ring homomorphism $R \longrightarrow H^0(G\pi R)$. In this section we shall show that given a set σ as in section 1 and an appropriate choice of \mathcal{A} , this is the noncommutative localization $R \longrightarrow H^0(G\pi R) = \sigma^{-1}R$ considered in section 1.

The first step is to give sufficient conditions for the Bousfield localization G to exist. We shall state a general existence theorem, and then narrow it to the case of interest. To state the general theorem, we need to remind the reader of compact objects in triangulated categories satisfying [TR5]. We recall the definitions.

Definition 3.1. A triangulated category \mathcal{C} is said to satisfy [TR5] if arbitrary coproducts exist in \mathcal{C} . □

Example 3.2. If R is an associative ring and $D(R)$ is its unbounded derived category, then $D(R)$ satisfies [TR5]. If a triangulated category satisfies [TR5], then coproducts of distinguished triangles are distinguished; see Proposition 1.2.1 and Remark 1.2.2 of [20]. □

Definition 3.3. Let \mathcal{C} be a triangulated category satisfying [TR5]. An object $c \in \mathcal{C}$ is called *compact* if the functor $\mathcal{C}(c, -)$ respects coproducts. Equivalently, c is compact if every map

$$c \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda$$

factors through a finite coproduct. □

Remark 3.4. In the derived category $D(R)$ of Example 3.2, the object $R \in D(R)$ is compact. By $R \in D(R)$ we mean the chain complex which is zero in all dimensions but 0, and is R in dimension 0. The proof that R is compact is the following. For any object $X \in D(R)$,

$$D(R)(R, X) = H^0(X) .$$

Since $H^0(-)$ is a functor commuting with coproducts, the compactness follows. All suspensions of a compact object are compact. □

It is useful to recall the following fact.

Lemma 3.5. *With the notation as in Remark 3.4, the category $D(R)$ contains an object R (the complex with R in degree zero, and zero in all other degrees). Any triangulated subcategory of $D(R)$, closed under coproducts and containing $R \in D(R)$, is all of $D(R)$.*

Proof. By Remark 3.4, the object R is compact. In the terminology of [20], R is an \aleph_0 -compact object in $D(R)$. Also, if for every $n \in \mathbb{Z}$ we have

$$D(R)(\Sigma^n R, X) = D(R)(R, \Sigma^{-n} X) = H^{-n}(X) = 0 ,$$

then X is acyclic and vanishes in $D(R)$. This makes the set $\{\Sigma^n R, n \in \mathbb{Z}\}$ an \aleph_0 -compact generating set for $D(R)$, as in Definition 8.1.6 of [20]. But then Theorem 8.3.3 loc. cit. tells us that

$$\langle \{\Sigma^n R, n \in \mathbb{Z}\} \rangle = D(R) ,$$

that is any triangulated subcategory of $D(R)$, closed under coproducts and containing $R \in D(R)$, is all of $D(R)$. □

Definition 3.6. Let \mathcal{C} be a triangulated category satisfying [TR5]. The subcategory $\mathcal{C}^c \subset \mathcal{C}$ is defined to be the full subcategory of all compact objects in \mathcal{C} .

□

Remark 3.7. The subcategory $\mathcal{C}^c \subset \mathcal{C}$ is always a thick subcategory.

□

Example 3.8. If R is an associative ring and $\mathcal{B} = D(R)$ is its unbounded derived category, then \mathcal{B}^c turns out to be the full subcategory of all objects isomorphic in $D(R) = \mathcal{B}$ to bounded complexes of f.g. projective R -modules. In the notation of the representation theorists, what we have been denoting $D(R)$ would be written $\mathcal{B} = D(R\text{-Mod})$, and \mathcal{B}^c is $D^b(R\text{-proj})$. The proof that $D^b(R\text{-proj})$ is precisely \mathcal{B}^c goes as follows. It is clear that $D^b(R\text{-proj})$ is a triangulated subcategory of \mathcal{B} , and Proposition 3.4 in [5] proves that it is also closed under direct summands. That is, $D^b(R\text{-proj})$ is a thick subcategory of \mathcal{B} . Example 1.13 of [21] shows that every object in $D^b(R\text{-proj})$ is compact; that is,

$$D^b(R\text{-proj}) \subset \mathcal{B}^c.$$

But R is an object of $D^b(R\text{-proj})$, and by Lemma 3.5 any triangulated category of \mathcal{B} closed under coproducts and containing R is all of \mathcal{B} . The fact that the inclusion

$$D^b(R\text{-proj}) \subset \mathcal{B}^c$$

is an equality now follows from Lemma 4.4.5 of [20], more precisely from the case where $\alpha = \beta = \aleph_0$ in that Lemma.

In the rest of this article, to keep the notation as simple as possible, we write $\mathcal{B} = D(R)$ for $D(R\text{-Mod})$, and $\mathcal{B}^c = D^c(R)$ for $D^b(R\text{-proj})$.

□

Next we state the main existence theorem for Bousfield localizations. The theorem gives sufficient conditions for the existence of a right adjoint G to the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/A$. It also tells us how the subcategories of compact objects, that is \mathcal{A}^c , \mathcal{B}^c and \mathcal{C}^c , are related to each other.

Theorem 3.9. *Let \mathcal{B} be a triangulated category satisfying [TR5]. Suppose there are two sets of compact objects $A, B \subset \mathcal{B}^c$ so that*

- (i) *Every triangulated subcategory of \mathcal{B} , closed under coproducts containing $B \subset \mathcal{B}^c$, must equal all of \mathcal{B} .*
- (ii) *The smallest triangulated subcategory of \mathcal{B} , closed under coproducts and containing $A \subset \mathcal{B}^c$, will be denoted A .*

We have an inclusion $A \subset \mathcal{B}$. Let $\mathcal{C} = \mathcal{B}/A$ be the quotient. The following is true

3.9.1. *A Bousfield localization functor exists for the pair $A \subset \mathcal{B}$. If G is right adjoint to the natural projection $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/A$, then G respects coproducts.*

3.9.2. *The categories \mathcal{A}^c , \mathcal{B}^c and \mathcal{C}^c are all essentially small.*

3.9.3. *$\mathcal{A}^c = \mathcal{A} \cap \mathcal{B}^c$. Furthermore, \mathcal{A}^c can also be described as the smallest thick subcategory of \mathcal{B}^c containing A .*

3.9.4. *The natural map $\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}$ is fully faithful, and factors through $\mathcal{C}^c \subset \mathcal{C}$. The functor*

$$\pi^c : \mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c$$

is almost an equivalence of categories. It is fully faithful by the above, and every object of \mathcal{C}^c is a direct summand of an object in the image of π^c .

Proof. For the proof of 3.9.1, see for example Lemma 1.7, Remark 1.8 and Proposition 1.9 in [21]. The proof of 3.9.4 may be found in Theorem 2.1 loc. cit.

For 3.9.3, observe that the inclusion $\mathcal{A} \cap \mathcal{B}^c \subset \mathcal{A}^c$ is obvious. An object of \mathcal{A} which is compact as an object in the larger \mathcal{B} must be compact in \mathcal{A} . The reverse inclusion, $\mathcal{A}^c \subset \mathcal{A} \cap \mathcal{B}^c$, may be found in Lemma 2.2 of [21], where it is also proved that \mathcal{A}^c is the smallest thick subcategory containing A .

Finally, for 3.9.2 we need to show that \mathcal{A}^c , \mathcal{B}^c and \mathcal{C}^c are all essentially small. For \mathcal{A}^c , this is obvious by 3.9.2; after all, \mathcal{A}^c is obtained from the set A of objects by completing with respect to triangles and direct summands. For \mathcal{B} , we deduce it from the previous case by choosing $A = B$, and therefore $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}^c = \mathcal{B}^c$. Now we know that \mathcal{B}^c and \mathcal{A}^c are essentially small. Hence so is $\mathcal{B}^c/\mathcal{A}^c$, and \mathcal{C}^c is obtained from it by splitting idempotents. Therefore \mathcal{C}^c is also essentially small. \square

Example 3.10. The case of most interest to us here is where $\mathcal{B} = D(R)$ is the derived category of the ring R ; we remind the reader, this means the derived category of all chain complexes of left R -modules. By Lemma 3.5, we know that we can let the set B be $B = \{R\}$. Every triangulated subcategory of \mathcal{B} , closed under coproducts and containing $B \subset \mathcal{B}^c$, is all of \mathcal{B} .

What Theorem 3.9 tells us is that we may choose any set of objects $A \subset \mathcal{B}^c$, and all the statements of the theorem hold. Let us begin by specialising to the case we wish to study. \square

Notation 3.11. Suppose that we are given a ring R . Let $\mathcal{B} = D(R)$ be the derived category of chain complexes of left R -modules. Let σ be a set of objects in $\mathcal{B}^c = D^c(R)$ of the form

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

That is, there exists an integer ℓ so that the only non-zero terms in the complex occur in dimensions ℓ and $\ell + 1$. And the R -modules c^ℓ and $c^{\ell+1}$ are both f.g. and projective. Note that we shall freely confuse the set of complexes σ with the set of maps

$$c^\ell \longrightarrow c^{\ell+1}.$$

Both sets will be called σ . We henceforth apply Theorem 3.9 to $\mathcal{B} = D(R)$ with $A = \sigma$ and $B = \{R\}$. Let $\mathcal{A} = D(R, \sigma)$ be the smallest triangulated subcategory of \mathcal{B} which is closed under coproducts and contains $A = \sigma \subset \mathcal{B}^c$. We shall write the full subcategory of $\mathcal{B}^c = D^c(R)$ consisting of the compact objects in \mathcal{A} as

$$\mathcal{A}^c = D^c(R, \sigma) .$$

With $\mathcal{C} = \mathcal{B}/\mathcal{A}$ as in Theorem 3.9, the functor $\pi : \mathcal{B} \longrightarrow \mathcal{C}$ has a right adjoint $G : \mathcal{C} \longrightarrow \mathcal{B}$. □

It is now time to recall another well-known fact about Bousfield localization. We begin with a definition.

Definition 3.12. Let \mathcal{B} be a triangulated category satisfying [TR5]. Let σ be a set of objects in \mathcal{B} . An object X is called σ -local if, for every object $s \in \sigma$ and every integer $n \in \mathbb{Z}$, $\mathcal{B}(s, \Sigma^n X) = 0$. □

Remark 3.13. The full subcategory of all σ -local objects is triangulated. □

Lemma 3.14. *Let the notation be as in 3.11. That is, $\mathcal{B} = D(R)$, σ is a set of objects in \mathcal{B}^c , and $\mathcal{A} = D(R, \sigma)$ is the smallest triangulated subcategory of \mathcal{B} containing σ and closed under coproducts. Suppose $x \in \mathcal{B}$ is σ -local. Then the unit of adjunction*

$$\eta_x : x \longrightarrow G\pi x$$

is an isomorphism.

Proof. Fix a σ -local object $x \in \mathcal{B}$. Consider the full subcategory $\mathcal{T} \subset \mathcal{B}$ of all objects $t \in \mathcal{B}$ so that, for every $n \in \mathbb{Z}$,

$$\mathcal{B}(t, \Sigma^n x) = \mathcal{B}(\Sigma^{-n} t, x) = 0 .$$

Because x is σ -local, \mathcal{T} must contain σ . But from its definition, it is clear that \mathcal{T} is closed under coproducts and triangles. Hence it must contain \mathcal{A} . We conclude that, for every object $a \in \mathcal{A}$ and any integer $n \in \mathbb{Z}$, $\mathcal{B}(\Sigma^n a, x) = 0$.

This means that, in the notation of Definition 9.1.10 of [20], x lies in ${}^\perp \mathcal{A}$. By Corollary 9.1.9 loc. cit, the unit of adjunction

$$\eta_x : x \longrightarrow G\pi x$$

is an isomorphism. □

Remark 3.15. The next lemmas will make use of the standard t -structure on $\mathcal{B} = D(R)$. The reader will find an excellent exposition of this topic in Chapter 1 of [3]. We recall that the full subcategory $\mathcal{B}^{\leq n}$ is defined by

$$\mathrm{Ob}(\mathcal{B}^{\leq n}) = \{X \in \mathrm{Ob}(\mathcal{B}) \mid H^r(X) = 0 \text{ for all } r > n\}.$$

□

Lemma 3.16. *As in Notation 3.11, let $\mathcal{B} = D(R)$, let σ be a set of objects in $\mathcal{B}^c = D^c(R)$ of the form*

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

If x is a σ -local object, then so are its t -structure truncations $x^{\leq n}$ and $x^{\geq n}$.

Proof. Pick a σ -local object x and an integer $n \in \mathbb{Z}$; without loss of generality we may assume $n = 0$. Let us begin by proving that $x^{\geq 0}$ is σ -local. Take any $s \in \sigma$, that is a chain complex

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

and any integer $m \in \mathbb{Z}$. By Definition 3.12, it suffices to show that any map $\Sigma^{-m}s \longrightarrow x^{\geq 0}$ vanishes. That is, we must prove that any map from

$$\dots \longrightarrow 0 \longrightarrow c^{\ell+m} \longrightarrow c^{\ell+m+1} \longrightarrow 0 \longrightarrow \dots$$

to $x^{\geq 0}$ must vanish in \mathcal{B} . If $\ell+m+1 < 0$, then there is no problem. We have $\Sigma^{-m}s \in \mathcal{B}^{<0}$ while $x^{\geq 0} \in \mathcal{B}^{\geq 0}$, and hence any map $\Sigma^{-m}s \longrightarrow x^{\geq 0}$ must vanish. Assume therefore that $\ell+m+1 \geq 0$.

We have a triangle

$$x^{<0} \longrightarrow x \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{<0}.$$

The composite

$$\Sigma^{-m}s \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{<0}$$

is a map from a bounded above complex of projectives $\Sigma^{-m}s$ to some object in $\mathcal{B} = D(R)$, and hence it is represented by a chain map. But the chain complex $\Sigma^{-m}s$ lives in degrees $\ell+m$ and $\ell+m+1$, both of which are ≥ -1 , while the complex $\Sigma x^{<0}$ lies in $\mathcal{B}^{\leq -2}$. Hence the map vanishes. From the triangle we deduce that the map $\Sigma^{-m}s \longrightarrow x^{\geq 0}$ must factor as

$$\Sigma^{-m}s \longrightarrow x \longrightarrow x^{\geq 0}.$$

But x is σ -local by hypothesis, and hence any map $\Sigma^{-m}s \longrightarrow x$ vanishes.

This proves that $x^{\geq 0}$ is σ -local. In the triangle

$$x^{<0} \longrightarrow x \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{<0}.$$

we now know that both x and $x^{\geq 0}$ are σ -local, and Remark 3.13 permits us to deduce that $x^{<0}$ is also σ -local. □

Lemma 3.17. *As in Notation 3.11, let $\mathcal{B} = D(R)$, let σ be a set of objects in $\mathcal{B}^c = D^c(R)$ of the form*

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

and let $\mathcal{A} = D(R, \sigma)$ be the smallest triangulated subcategory of \mathcal{B} containing σ and closed under coproducts. We assert that if $x \in \mathcal{B}^{\leq n}$, then so is $G\pi x$.

Proof. We may assume without loss that $n = 0$. Pick any $x \in \mathcal{B}^{\leq 0}$. For any $s \in \sigma$ and any $n \in \mathbb{Z}$, we have by adjunction

$$\mathcal{B}(\Sigma^{-m}s, G\pi x) = \mathcal{C}(\pi\Sigma^{-m}s, \pi x)$$

which vanishes since $\pi\Sigma^{-m}s = 0$. Hence $G\pi x$ is σ -local, and by Lemma 3.16 so is $\{G\pi x\}^{\leq 0}$.

Now the unit of adjunction $\eta_x : x \longrightarrow G\pi x$ is a map from an object $x \in \mathcal{B}^{\leq 0}$, and therefore factors (uniquely) as

$$x \xrightarrow{\alpha} \{G\pi x\}^{\leq 0} \xrightarrow{f} G\pi x.$$

Since $\{G\pi x\}^{\leq 0}$ is σ -local, Lemma 3.14 coupled with 2.6.3 tell us that the map α factors (uniquely) as

$$x \xrightarrow{\eta_x} G\pi x \xrightarrow{g} \{G\pi x\}^{\leq 0}.$$

The composite

$$x \xrightarrow{\eta_x} G\pi x \xrightarrow{g} \{G\pi x\}^{\leq 0} \xrightarrow{f} G\pi x$$

is η_x , and the uniqueness statement in 2.6.3 says that $fg : G\pi x \longrightarrow G\pi x$ is the identity. The identity factors through an object in $\mathcal{B}^{\leq 0}$.

But then the identity must induce the zero map on all $H^n(G\pi x)$ with $n > 0$. That is, for $n > 0$ we have $H^n(G\pi x) = 0$. In other words, $G\pi x \in \mathcal{B}^{\leq 0}$. \square

Remark 3.18. The *derived tensor product* is defined for any $R - R$ bimodule S to be the triangulated, coproduct-preserving functor

$$D(R) \longrightarrow D(R) ; X \mapsto S^L \otimes_R X = S \otimes_R P$$

with P a sufficiently nice projective R -module chain complex quasi-isomorphic to X . The existence of this functor may be found, for example, in Theorem 2.14 of Bökstedt and Neeman [5]. The following are sufficiently nice :

- (a) bounded above chain complexes of projectives,
- (b) coproducts of the above,
- (c) mapping cones of chain maps between the above.

In this article, we consider tensor products both in the category of modules and in the derived category. We try to be careful to distinguish them in the notation.

\square

Proposition 3.19. *As in Notation 3.11, let $\mathcal{B} = D(R)$, let σ be a set of objects in $\mathcal{B}^c = D^c(R)$ of the form*

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

and let $\mathcal{A} = \mathcal{D}(R, \sigma) \subset \mathcal{B}$ be the smallest triangulated subcategory containing σ and closed under coproducts.

By Lemma 2.8, we have a ring homomorphism $H^0(\eta_R) : R \longrightarrow S$, with $S = H^0(G\pi R)$. Let $g : R \longrightarrow T$ be a ring homomorphism such that for every $s \in \sigma \subset \mathcal{B}^c$ we have $T^L \otimes_R s = 0$. Then g factors uniquely through the ring homomorphism $H^0(\eta_R) : R \longrightarrow S$, as below :

$$\begin{array}{ccc} & S & \\ H^0(\eta_R) \nearrow & & \searrow \exists! f \\ R & \xrightarrow{g} & T \end{array}$$

Proof. For any object $X \in D(R) = \mathcal{B}$, we have

$$D(R)(X, T) = D(T)(T^L \otimes_R X, T) .$$

By hypothesis, $T^L \otimes_R s$ vanishes for all $s \in \sigma$. Since

$$D(R)(\Sigma^r s, T) = D(T)(T^L \otimes_R \Sigma^r s, T) = 0$$

for all $r \in \mathbb{Z}$ and all $s \in \sigma$, it follows that $T \in D(R)$ is σ -local, and by Lemma 3.14 the unit of adjunction

$$\eta_T : T \longrightarrow G\pi T$$

is an isomorphism. But then

$$\begin{aligned} \mathcal{B}(R, T) &= \mathcal{B}(R, G\pi T) && \text{because } \eta_T \text{ is an isomorphism} \\ &= \mathcal{B}(G\pi R, G\pi T) && \text{by 2.6.3.} \end{aligned}$$

This concretely translates to saying that the map $R \longrightarrow T = G\pi T$ factors uniquely through $\eta_R : R \longrightarrow G\pi R$. Applying H^0 to this, we have a factorization

$$R \xrightarrow{H^0(\eta_R)} H^0(G\pi R) = S \xrightarrow{f} T .$$

To see that the factorization is unique note that, by Lemma 3.17, we know that $G\pi R \in \mathcal{B}^{\leq 0}$. The homomorphism $R \longrightarrow S$ therefore factors, in the derived category $D(R) = \mathcal{B}$, as the composite

$$R \xrightarrow{\eta_R} G\pi R \longrightarrow \{G\pi R\}^{\geq 0} = H^0(G\pi R) = S .$$

Given any morphism of R -modules $f' : S \longrightarrow T$ we can form the composite

$$R \xrightarrow{\eta_R} G\pi R \longrightarrow \{G\pi R\}^{\geq 0} = H^0(G\pi R) \xrightarrow{f'} T .$$

If this composite agrees with g , then by the uniqueness of the factorization through $R \longrightarrow G\pi R$ above, the composites

$$\begin{aligned} G\pi R &\longrightarrow \{G\pi R\}^{\geq 0} \xrightarrow{f} T \\ G\pi R &\longrightarrow \{G\pi R\}^{\geq 0} \xrightarrow{f'} T \end{aligned}$$

must agree. But $T \in \mathcal{B}^{\geq 0}$. Therefore any map $G\pi R \longrightarrow T$ factors uniquely through the t -structure truncation

$$G\pi R \longrightarrow \{G\pi R\}^{\geq 0} \longrightarrow T .$$

Hence $f = f'$. We have proved that there exists a unique map f of left R -modules making the triangle commute

$$\begin{array}{ccc} & S & \\ H^0(\eta_R) \nearrow & & \searrow \exists! f \\ R & \xrightarrow{g} & T \end{array}$$

Now consider the functor

$$D(R) \longrightarrow D(T) ; X \mapsto T^L \otimes_R X .$$

By the hypothesis on T , this functor annihilates all complexes $s \in \sigma$. Hence it annihilates the subcategory $\mathcal{A} = D(R, \sigma) \subset \mathcal{B} = D(R)$. The functor factors as

$$D(R) \xrightarrow{\pi} \mathcal{C} = D(R)/D(R, \sigma) \longrightarrow D(T)$$

Hence we have ring homomorphisms

$$\mathrm{End}_{D(R)}(R) \xrightarrow{\alpha} \mathrm{End}_{\mathcal{C}}(\pi R) \xrightarrow{\beta} \mathrm{End}_{D(T)}(T) .$$

In Lemma 2.8, we checked that α agrees with the homomorphism $H^0(\eta_R) : R \longrightarrow H^0(G\pi R) = S$, while from the definition of the functor $D(R) \longrightarrow D(T)$ by tensor products, the composite $\beta\alpha$ is nothing other than the given map $g : R \longrightarrow T$. It follows that there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} & S & \\ H^0(\eta_R) \nearrow & & \searrow \exists f' \\ R & \xrightarrow{g} & T \end{array}$$

But we already know that f' is unique, even as a map of R -modules. There is a unique ring homomorphism rendering commutative the triangle. \square

Lemma 3.20. *The ring $S = H^0(G\pi R)$ satisfies the property that, for all $s : P \longrightarrow Q$ in σ the induced S -module morphism $1 \otimes s : S \otimes_R P \longrightarrow S \otimes_R Q$ is an isomorphism.*

Proof. By Lemma 3.17, $G\pi R \in \mathcal{B}^{\leq 0}$. Hence $S = H^0(G\pi R) = \{G\pi R\}^{\geq 0}$. By Lemma 3.16, $\{G\pi R\}^{\geq 0}$ is σ -local.

But then, for all $\{c^\ell \longrightarrow c^{\ell+1}\} \in \sigma$, the map

$$\mathrm{Hom}_R(c^{\ell+1}, S) \longrightarrow \mathrm{Hom}_R(c^\ell, S)$$

is an isomorphism. Applying the functor $\mathrm{Hom}_S(-, S)$ to this isomorphism, and recalling that

$$\mathrm{Hom}_S[\mathrm{Hom}_R(A, S), S] = S \otimes_R A,$$

we deduce that $S \otimes_R c^\ell \longrightarrow S \otimes_R c^{\ell+1}$ is an isomorphism. \square

Theorem 3.21. *With the notation as above, the ring homomorphism $R \longrightarrow S = H^0(G\pi R)$ satisfies the universal property of $R \longrightarrow \sigma^{-1}R$.*

Proof. Lemma 3.20 and Proposition 3.19. \square

Notation 3.22. We have now proved that, with R a ring, $\mathcal{B} = D(R)$ its derived category, and $\sigma \subset \mathcal{B}^c$, $\mathcal{A} = D(R, \sigma) \subset \mathcal{B}$ and $\mathcal{C} = \mathcal{B}/\mathcal{A}$ as in Notation 3.11, the ring homomorphism $\eta_R : R \longrightarrow H^0(G\pi R)$ satisfies Cohn's universal property. From now on, we shall freely confuse $H^0(G\pi R) = \sigma^{-1}R$. \square

The object $G\pi R$ in $D(R)$ is isomorphic to a chain complex $E(R)$ of free R -modules bounded above by 0. We shall prove that $\sigma^{-1}E(R)$ is isomorphic to $\sigma^{-1}R$. In fact, Vogel [32] gave a direct construction for $E(R)$ as the direct limit of successive mapping cones. This construction is very reminiscent of Bousfield's original proof of the existence of a Bousfield localization; see Bousfield's [6] and [7]. By now, of course, there are other proofs of the existence of G .

Remark 3.23. The universal property of $\sigma^{-1}R$ is self-dual in the following sense. We are given a set σ of morphisms of f.g. projective left R -modules. If $s : P \longrightarrow Q$ is an element of σ , we can look at the dual map $s^* : Q^* \longrightarrow P^*$. Here, $X^* = \mathrm{Hom}_R(X, R)$. To say that

$$1 \otimes s : S \otimes_R P \longrightarrow S \otimes_R Q$$

is an isomorphism is equivalent to saying that

$$s^* \otimes 1 : Q^* \otimes_R S \longrightarrow P^* \otimes_R S$$

is an isomorphism. In other words, we can proceed to do the entire construction of $\sigma^{-1}R$ in terms of right R -modules, just by replacing σ by the set σ^* of maps $s^* : Q^* \longrightarrow P^*$. Every theorem we prove has a dual version for right modules. \square

Proposition 3.24. *Let M be a right R -module so that, for all $s : P \rightarrow Q$ in σ , $1 \otimes_R s : M \otimes_R P \rightarrow M \otimes_R Q$ is an isomorphism. Then the map $1^L \otimes_R \eta_R : M \rightarrow M^L \otimes_R G\pi R$ is an isomorphism.*

Proof. Form the triangle

$$a \longrightarrow R \xrightarrow{\eta_R} G\pi R \longrightarrow \Sigma a .$$

By 2.6.5, $a \in \mathcal{A}$. Now the set of objects $x \in \mathcal{B} = D(R)$ such that $M^L \otimes_R x$ vanishes is a triangulated category containing σ , and closed under coproducts. It must contain all of \mathcal{A} . Therefore $M^L \otimes_R a = 0$. The triangle

$$M^L \otimes_R a \longrightarrow M \xrightarrow{M^L \otimes_R \eta_R} M^L \otimes_R G\pi R \longrightarrow M^L \otimes_R \Sigma a$$

tells us that $M^L \otimes_R \eta_R$ must be an isomorphism. \square

Remark 3.25. Dually as in Remark 3.23, given any left module M so that, for any $s^* : Q^* \rightarrow P^*$ in σ^* , the map $s^* \otimes 1$ is an isomorphism, then $\eta'_R{}^L \otimes_R 1 : M \rightarrow G'\pi'R^L \otimes_R M$ is an isomorphism. [Here, G' , π' and η' are the duals of G , π and η in the derived category of right R -modules]. Note that

$$P^{*L} \otimes_R M = \text{Hom}_R(P, R) \otimes_R M = \text{Hom}_R(P, M) .$$

We deduce that $s^* \otimes 1_M$ is an isomorphism if and only if $\mathcal{B}(\Sigma^n s, M) = 0$ for all $n \in \mathbb{Z}$. The M 's for which $s^* \otimes 1_M$ are isomorphisms whenever $s \in \sigma$ are precisely the σ -local objects, as in Definition 3.12. Summarising: whenever M is a left R -module, which as an object of $\mathcal{B} = D(R)$ happens to be σ -local, then the dual of Proposition 3.24 asserts that $\eta'_R{}^L \otimes_R 1 : M \rightarrow G'\pi'R^L \otimes_R M$ is an isomorphism. \square

Proposition 3.26. *Let M be any right R -module so that, for all $s : P \rightarrow Q$ in σ , $1 \otimes_R s : M \otimes_R P \rightarrow M \otimes_R Q$ is an isomorphism. Then the map $1 \otimes H^0(\eta_R) : M \rightarrow M \otimes_R \{\sigma^{-1}R\}$ is an isomorphism. Furthermore,*

$$\text{Tor}_1^R(M, \sigma^{-1}R) = 0 .$$

Proof. There is a spectral sequence computing the cohomology of $M^L \otimes_R G\pi R$. The E_2 term is

$$E_2^{i,j} = \text{Tor}_{-i}^R(M, H^j(G\pi R)) .$$

By Lemma 3.16, $H^j(G\pi R) = 0$ for all $j > 0$. And Tor_{-i}^R vanishes for $i > 0$. This spectral sequence is concentrated in negative degrees. Because all the differentials in and out of the following terms vanish, we conclude that, in the spectral sequence, $E_2^{0,0} = E_\infty^{0,0}$ and $E_2^{-1,0} = E_\infty^{-1,0}$.

By Proposition 3.24, we have that $M^L \otimes_R G\pi R = M$. The above spectral sequence converges to $H^{i+j}(M)$, which is M if $i + j = 0$ and zero otherwise. We immediately conclude that

$$E_2^{0,0} = M \otimes_R \sigma^{-1}R = M$$

and

$$E_2^{-1,0} = \mathrm{Tor}_1^R(M, \sigma^{-1}R) = 0.$$

□

Corollary 3.27. *The natural multiplication map $\sigma^{-1}R \otimes_R \sigma^{-1}R \longrightarrow \sigma^{-1}R$ is an isomorphism, and*

$$\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0.$$

Proof. In Proposition 3.26, put $M = \sigma^{-1}R$. We immediately deduce that $\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0$. We also have that the map

$$1 \otimes H^0(\eta_R) : \sigma^{-1}R \longrightarrow \{\sigma^{-1}R\} \otimes_R \{\sigma^{-1}R\}$$

is an isomorphism. But the composite

$$\sigma^{-1}R \xrightarrow{1 \otimes H^0(\eta_R)} \{\sigma^{-1}R\} \otimes_R \{\sigma^{-1}R\} \xrightarrow{\text{multiplication}} \sigma^{-1}R$$

is clearly the identity, and hence the multiplication map must be the two-sided inverse of the invertible map $1 \otimes H^0(\eta_R)$. □

Remark 3.28. The result $\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0$ is due to Schofield [27], p.58.

□

Corollary 3.29. *Suppose M is a left R -module, which as an object of $\mathcal{B} = D(R)$ is σ -local (see Definition 3.12). Then the R -module structure of M extends, uniquely, to a $\sigma^{-1}R$ -module structure.*

Proof. The uniqueness of the extension is clear: to say that M is a left R -module is to give a homomorphism

$$R \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, M),$$

and the universal property of the noncommutative localization tells us that the factorization of this ring homomorphism through $\sigma^{-1}R$ is certainly unique, if it exists.

It remains to prove existence. By the dual of Proposition 3.26, the map

$$M = R \otimes_R M \xrightarrow{H^0(\eta_R) \otimes_R 1} \{\sigma^{-1}R\} \otimes_R M$$

is an isomorphism. Thus M is isomorphic, as a left R -module, to $\{\sigma^{-1}R\} \otimes_R M$, and $\{\sigma^{-1}R\} \otimes_R M$ is certainly a module over $\sigma^{-1}R$. □

In the remainder of this section, we want to relate $H^n(G\pi R)$ for $n < 0$ to $\mathrm{Tor}_*^R(\sigma^{-1}R, \sigma^{-1}R)$. First we prove a lemma.

Lemma 3.30. *Suppose $R \rightarrow S$ is a ring homomorphism such that the multiplication map $S \otimes_R S \rightarrow S$ is an isomorphism. Suppose also that for some $n \geq 1$*

$$\mathrm{Tor}_i^R(S, S) = 0 \quad (1 \leq i \leq n) .$$

Then for every S -module M , we have

3.30.1. *The multiplication map $S \otimes_R M \rightarrow M$ is an isomorphism.*

3.30.2. $\mathrm{Tor}_i^R(S, M) = 0$ *for all $1 \leq i \leq n$.*

Proof. Choose a resolution of M by free S -modules

$$\cdots \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow M \longrightarrow 0 ,$$

and a resolution of S by free R -modules

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow S \longrightarrow 0 .$$

The tensor product $P \otimes Q$ gives a double complex whose cohomology computes $\mathrm{Tor}_{-i-j}^R(S, M)$. But there is a spectral sequence for it, whose E_1 term is

$$E_1^{i,j} = \mathrm{Tor}_{-j}^R(S, Q^i)$$

Now $E_1^{i,0} = S \otimes_R Q^i = Q^i$, since Q^i is free and, by hypothesis, $S \otimes_R S \rightarrow S$ is an isomorphism. In E_2 , we have

$$E_2^{i,0} = \begin{cases} M & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

But by hypothesis, we also have $\mathrm{Tor}_{-j}^R(S, S) = 0$, for all $1 \leq -j \leq n$, and since Q^i are free, this gives $\mathrm{Tor}_{-j}^R(S, Q^i) = 0$, for all i and for all $1 \leq -j \leq n$. In other words, $E_1^{i,j} = 0$ if $1 \leq -j \leq n$, and hence $E_2^{i,j} = 0$ if either $j = 0, i \neq 0$, or if $1 \leq -j \leq n$. The assertions of the Lemma immediately follow. \square

Corollary 3.31. *Let the notation be as in Notation 3.11. Suppose $\mathrm{Tor}_i^R(\sigma^{-1}R, \sigma^{-1}R) = 0$, for all $1 \leq i \leq n$. Then for all $1 \leq i \leq n-1$ we have $H^{-i}(G\pi R) = 0$, and*

$$\mathrm{Tor}_{n+1}^R(\sigma^{-1}R, \sigma^{-1}R) = H^{-n}(G\pi R) .$$

Proof. The proof is a slightly more sophisticated computation with the same spectral sequence we saw in Proposition 3.26. Recall that we have a spectral sequence whose E_2 term is

$$E_2^{i,j} = \mathrm{Tor}_{-i}^R(\sigma^{-1}R, H^j(G\pi R)) ,$$

which converges to $H^{i+j}(\sigma^{-1}R)$. Now we know that $G\pi R$ is σ -local. By Lemma 3.16 so are its t -structure truncations $H^j(G\pi R) = \left\{ [G\pi R]^{\leq j} \right\}^{\geq j}$. Corollary 3.29 now tells us that $H^j(G\pi R)$ are all left $\sigma^{-1}R$ -modules. Lemma 3.30 now applies, and we deduce that if $1 \leq -i \leq n$ then $E_2^{i,j} = 0$. This forces the differential

$$E_2^{-i-1,0} \longrightarrow E_2^{0,-i}$$

to be an isomorphism, for all $1 \leq i \leq n$. For $1 \leq i \leq n-1$ we read off that $H^{-i}(G\pi R) = 0$. For $i = n$, we deduce that

$$\mathrm{Tor}_{n+1}^R(\sigma^{-1}R, \sigma^{-1}R) = H^{-n}(G\pi R) .$$

□

4. A BOUND ON THE LENGTH OF COMPLEXES IN $D^c(R, \sigma)$

Notation 4.1. Our notation stays as in Notation 3.11. $D(R) = \mathcal{B}$ is the derived category of a ring R , and we are given a set σ of maps of f.g. projective R -modules. The category $\mathcal{A} = D(R, \sigma)$ is the smallest triangulated subcategory of \mathcal{B} containing σ and closed under coproducts. Let $\mathcal{C} = \mathcal{B}/\mathcal{A}$. Let $\pi : \mathcal{B} \rightarrow \mathcal{C}$ be the projection, $G : \mathcal{C} \rightarrow \mathcal{B}$ the fully faithful right adjoint. Identify $R \rightarrow \sigma^{-1}R$ with $R \rightarrow H^0(G\pi R)$.

□

For any f.g. projective R -modules M, N a morphism $\pi M \rightarrow \pi N$ in \mathcal{C} is an equivalence class of diagrams

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N$$

with α a morphism in \mathcal{B} which becomes an isomorphism in \mathcal{C} . Later in the article, we will need to have bounds on the length of Y . In this section we carry out the preparatory technical work.

Definition 4.2. The full subcategory of all objects in $\mathcal{B} = D(R)$ which vanish outside the range $[m, n]$ will be denoted $\mathcal{B}[m, n]$. We allow m or n to be infinite; the categories $\mathcal{B}[m, \infty)$ and $\mathcal{B}(-\infty, n]$ have the obvious definitions.

□

Remark 4.3. The reader should note that the categories $\mathcal{B}[n, \infty)$ and $\mathcal{B}(-\infty, n]$ should not be confused with $\mathcal{B}^{\geq n}$ and $\mathcal{B}^{\leq n}$. It is true that every object in $\mathcal{B}^{\leq n}$ is *isomorphic in* \mathcal{B} to a chain complex

$$\cdots \rightarrow X^m \rightarrow X^{m+1} \rightarrow \cdots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

An isomorphism in $\mathcal{B} = D(R)$ is after all just a homology isomorphism. For any object in $\mathcal{B}^{\leq n}$, there is an object in $\mathcal{B}(-\infty, n]$ homology isomorphic to it. But for once we want to have a name for the complexes which are actually supported on the interval $[m, n]$, not just isomorphic in \mathcal{B} to such objects.

□

Definition 4.4. The category \mathcal{S} will be the smallest full subcategory of \mathcal{B} such that

4.4.1. *Every suspension of every object in σ lies in \mathcal{S} . That is, \mathcal{S} contains all the complexes*

$$\cdots \rightarrow 0 \rightarrow c^{\ell+n} \rightarrow c^{\ell+n+1} \rightarrow 0 \rightarrow \cdots$$

4.4.2. Given any chain map of objects in \mathcal{S}

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & X^{i-1} & \xrightarrow{\partial} & X^i & \xrightarrow{\partial} & X^{i+1} & \xrightarrow{\partial} & \dots \\ & & f_{i-1} \downarrow & & f_i \downarrow & & f_{i+1} \downarrow & & \\ \dots & \xrightarrow{\partial} & Y^{i-1} & \xrightarrow{\partial} & Y^i & \xrightarrow{\partial} & Y^{i+1} & \xrightarrow{\partial} & \dots \end{array}$$

then the mapping cone

$$\dots \longrightarrow X^i \oplus Y^{i-1} \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_i & \partial \end{pmatrix}} X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_{i+1} & \partial \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \longrightarrow \dots$$

also lies in \mathcal{S} .

As in Remark 4.3, we mean equality of chain complexes, not homotopy equivalence. \square

Definition 4.5. The subcategories $\mathcal{S}[m, n]$ are defined as the intersection

$$\mathcal{S}[m, n] = \mathcal{S} \cap \mathcal{B}[m, n].$$

As in Definition 4.2, we allow m and n to be infinite. \square

Lemma 4.6. Suppose $n \in \mathbb{Z}$ is an integer. Then every object $Z \in \mathcal{S}$ can be expressed as a mapping cone on a chain map $Z_1 \longrightarrow Z_2$, as below

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial} & Z_1^{n-1} & \xrightarrow{\partial} & Z_1^n & \xrightarrow{\partial} & Z_1^{n+1} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & f_n \downarrow & & f_{n+1} \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & Z_2^n & \xrightarrow{\partial} & Z_2^{n+1} & \xrightarrow{\partial} & Z_2^{n+2} & \xrightarrow{\partial} & \dots \end{array}$$

that is, $Z_1 \in \mathcal{S}(-\infty, n+1]$ and $Z_2 \in \mathcal{S}[n, \infty)$.

Proof. Let \mathcal{T} be the full subcategory of \mathcal{S} containing the objects for which the assertion of the lemma holds. That is, an object $Z \in \mathcal{S}$ belongs to \mathcal{T} if and only if, for every $n \in \mathbb{Z}$, there exist $Z_1 \in \mathcal{S}(-\infty, n+1]$ and $Z_2 \in \mathcal{S}[n, \infty)$ and a chain map $Z_1 \longrightarrow Z_2$ so that Z is equal to the mapping cone. It suffices to prove that $\mathcal{T} = \mathcal{S}$, for which we need only show that any suspension of an object of σ lies in \mathcal{T} , and that mapping cones on maps in \mathcal{T} lie in \mathcal{T} .

Assume therefore that we are given a complex s below

$$\dots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \dots$$

which is some suspension of an object in σ . Choose any $n \in \mathbb{Z}$. If $n \leq \ell$, then $s \in \mathcal{S}[n, \infty)$, and s is the mapping cone of the chain map $0 \longrightarrow s$. If $n \geq \ell+1$, then $\Sigma^{-1}s \in \mathcal{S}(-\infty, n+1]$ and s is isomorphic to the mapping cone on the chain map $\Sigma^{-1}s \longrightarrow 0$. Either way, $s \in \mathcal{T}$.

Next suppose we are given two object X and Y in \mathcal{T} , and a chain map $f : X \rightarrow Y$. Let Z be the mapping cone of f . We need to show that Z is in \mathcal{T} . For every integer $n \in \mathbb{Z}$, we need to express Z as a mapping cone on a map of objects $Z_1 \rightarrow Z_2$, with $Z_1 \in \mathcal{S}(-\infty, n+1]$ and $Z_2 \in \mathcal{S}[n, \infty)$. Without loss of generality, assume $n = 0$.

Because $X \in \mathcal{T}$, we may express it as a mapping cone on a map $X_1 \rightarrow X_2$, with $X_1 \in \mathcal{S}(-\infty, 2]$ and $X_2 \in \mathcal{S}[1, \infty)$. Because $Y \in \mathcal{T}$, we may express it as the mapping cone on a map $Y_1 \rightarrow Y_2$, with $Y_1 \in \mathcal{S}(-\infty, 1]$ and $Y_2 \in \mathcal{S}[0, \infty)$. We have a diagram, where the rows are short exact sequences of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & \Sigma X_1 \longrightarrow 0 \\ & & & & \downarrow f & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & Y & \longrightarrow & \Sigma Y_1 \longrightarrow 0 \end{array}$$

The composite

$$\begin{array}{ccc} X_2 & \longrightarrow & X \\ & & \downarrow f \\ & & Y \longrightarrow \Sigma Y_1 \end{array}$$

is a chain map from $X_2 \in \mathcal{S}[1, \infty)$ to $\Sigma Y_1 \in \mathcal{S}(-\infty, 0]$, and therefore must vanish. It follows that we may complete to a commutative diagram of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & \Sigma X_1 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow \Sigma f_2 \\ 0 & \longrightarrow & Y_2 & \longrightarrow & Y & \longrightarrow & \Sigma Y_1 \longrightarrow 0 \end{array}$$

Let Z_1 be the mapping cone on $f_1 : X_2 \rightarrow Y_2$, and let Z_2 be the mapping cone on $f_2 : X \rightarrow Y$. Then $Z_1 \in \mathcal{S}(-\infty, 1]$ while $Z_2 \in \mathcal{S}[0, \infty)$. Furthermore, Z , which is the mapping cone on $f : X \rightarrow Y$, can also be expressed as a mapping cone on a map $Z_1 \rightarrow Z_2$. \square

Lemma 4.7. *Let $\tilde{\mathcal{S}}$ be the full subcategory of all objects in \mathcal{B} isomorphic to objects in \mathcal{S} . That is, any object of $\mathcal{B} = D(R)$ isomorphic to a chain complex in \mathcal{S} lies in $\tilde{\mathcal{S}}$. The subcategory $\tilde{\mathcal{S}} \subset \mathcal{B}$ is triangulated.*

Proof. The point is that the objects of \mathcal{S} are bounded chain complexes of projectives. Let $f : X \rightarrow Y$ be a morphism in $D(R)$, between objects in \mathcal{S} . Because X is a bounded-above complex of projectives, there is a chain map representing the morphism. The mapping cone on this chain map completes $f : X \rightarrow Y$ to a triangle, and lies in \mathcal{S} . Up to isomorphism in $\mathcal{B} = D(R)$, the third edge is unique. Therefore in any triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

we have $Z \in \tilde{\mathcal{S}}$. \square

Lemma 4.8. *The category \mathcal{S} is contained in \mathcal{A}^c . Furthermore, the inclusion is nearly an equality. Every object in \mathcal{A}^c is a direct summand of an object isomorphic in $D(R)$ to an object in \mathcal{S} .*

Proof. The inclusion $\mathcal{S} \subset \mathcal{A}^c$ is easy. The category \mathcal{A}^c contains σ and is closed under mapping cones, and \mathcal{S} is the smallest such.

Next observe that, by 3.9.3, the category \mathcal{A}^c is the smallest thick subcategory of \mathcal{B} containing σ , and hence \mathcal{A}^c is the smallest thick subcategory containing the triangulated subcategory $\tilde{\mathcal{S}}$ of Lemma 4.7. Now Corollary 4.5.12 of [20] tells us that, for every object $X \in \mathcal{A}^c$, the object $X \oplus \Sigma X$ lies in $\tilde{\mathcal{S}}$. In particular $X \in \mathcal{A}^c$ is a direct summand of an object isomorphic in \mathcal{B} to an object in \mathcal{S} . \square

Lemma 4.9. *The natural map $\mathcal{B}^c/\tilde{\mathcal{S}}^c \longrightarrow \mathcal{C}$ is fully faithful.*

Proof. By 3.9.4, the functor

$$\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c \subset \mathcal{C}$$

is fully faithful. Now by Lemma 4.8, the thick closure of $\tilde{\mathcal{S}}$ is all of \mathcal{A}^c , and hence $\mathcal{B}^c/\mathcal{A}^c = \mathcal{B}^c/\tilde{\mathcal{S}}^c$. \square

Remark 4.10. One way we shall use the results of this section is as follows. Suppose $x \longrightarrow y$ is a morphism in \mathcal{B}^c , which maps to zero in $\mathcal{C} = \mathcal{B}/\mathcal{A}$. By Lemma 4.9, the map vanishes already in $\mathcal{B}^c/\tilde{\mathcal{S}}$. By Lemma 2.1.16 of [20], $x \longrightarrow y$ must factor as $x \longrightarrow s \longrightarrow y$ with $s \in \mathcal{S}$. The results of this section will enable us to replace s by shorter complexes. \square

In section 12 we will need the following result

Proposition 4.11. *Let M and N be any f.g. projective R -modules, which we view as objects in the derived category $\mathcal{B} = D(R)$, concentrated in degree 0. Then any map in $\mathcal{C}(\pi M, \pi N)$ can be represented as $\alpha^{-1}\beta$, for some α, β as below*

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \longrightarrow \Sigma X$$

and X may be chosen to lie in $\mathcal{S}[0, 1]$.

Proof. Both M and N are assumed to be objects of \mathcal{B}^c , and by Lemma 4.9 the map $\mathcal{B}/\tilde{\mathcal{S}}^c \longrightarrow \mathcal{C}$ is fully faithful. Hence

$$\mathcal{C}(\pi M, \pi N) = \{\mathcal{B}^c/\tilde{\mathcal{S}}^c\}(M, N).$$

Any morphism may be represented by a diagram

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N$$

so that in the triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \longrightarrow \Sigma X$$

X may be chosen to lie in \mathcal{S} .

By Lemma 4.6, there exists a triangle in \mathcal{A}^c

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{S}[1, \infty)$ and $X'' \in \mathcal{S}(-\infty, 1]$. The composite $X' \longrightarrow X \longrightarrow N$ is a map from $X' \in \mathcal{S}[1, \infty)$ to $N \in \mathcal{S}[0, 0]$, which must vanish. Hence we have that $X \longrightarrow N$ factors as $X \longrightarrow X'' \longrightarrow N$. We complete to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & N & \xrightarrow{\gamma\alpha} & Y'' & \longrightarrow & \Sigma X'' \end{array}$$

and another representative of our morphism is the diagram

$$M \xrightarrow{\gamma\beta} Y'' \xleftarrow{\gamma\alpha} N$$

We may, on replacing Y by Y'' , assume $X \in \mathcal{S}(-\infty, 1]$.

Applying Lemma 4.6 again, we have that any $X \in \mathcal{S}(-\infty, 1]$ admits a triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{S}[0, 1]$ and $X'' \in \mathcal{S}(-\infty, 0]$. Form the octahedron

$$\begin{array}{ccccccc} X' & \longrightarrow & N & \xrightarrow{\alpha'} & Y' & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\ & & & & \downarrow & & \downarrow \\ & & & & \Sigma X'' & \xrightarrow{1} & \Sigma X'' \end{array}$$

The composite $M \longrightarrow Y \longrightarrow \Sigma X''$ is a map from $M \in \mathcal{S}[0, 0]$ to $\Sigma X'' \in \mathcal{S}(\infty, -1]$, and must vanish. The map $\beta : M \longrightarrow Y$ therefore factors as $M \xrightarrow{\beta'} Y' \xrightarrow{\gamma} Y$, and our morphism in \mathcal{C} has a representative

$$M \xrightarrow{\beta'} Y' \xleftarrow{\alpha'} N$$

so that in the triangle

$$X' \longrightarrow N \xrightarrow{\alpha'} Y' \longrightarrow \Sigma X'$$

X' may be chosen to lie in $\mathcal{S}[0, 1]$. □

5. THE FUNCTOR $T : (D(R)/D(R, \sigma))^c \rightarrow D^c(\sigma^{-1}R)$

Let $R \rightarrow S$ be a ring homomorphism. There is a triangulated functor $D(R) \rightarrow D(S)$, taking $X \in D(R)$ to $S^L \otimes_R X$. In this section, we shall study this functor in the case where R is a ring, and $S = \sigma^{-1}R$ is the noncommutative localization of R inverting σ .

Proposition 5.1. *The functor*

$$\mathcal{B} = D(R) \longrightarrow \mathcal{D} = D(\sigma^{-1}R) ; X \mapsto \{\sigma^{-1}R\}^L \otimes_R X$$

factors uniquely (up to canonical natural isomorphism) through $\mathcal{B} \rightarrow \mathcal{C} = D(R)/D(R, \sigma)$. The unique factorization will be written as

$$\mathcal{B} \xrightarrow{\pi} \mathcal{C} \xrightarrow{T} \mathcal{D} .$$

The functor $T : \mathcal{C} \rightarrow \mathcal{D}$ takes $\mathcal{C}^c \subset \mathcal{C}$ to $\mathcal{D}^c = D^c(\sigma^{-1}R) \subset \mathcal{D} = D(\sigma^{-1}R)$.

Proof. Let $c^\ell \rightarrow c^{\ell+1}$ be a map in σ . Tensoring with $\sigma^{-1}R$ takes it to an isomorphism. Hence tensoring with $\sigma^{-1}R$ takes the chain complex

$$\cdots \rightarrow 0 \rightarrow c^\ell \rightarrow c^{\ell+1} \rightarrow 0 \rightarrow \cdots$$

to an acyclic complex. Therefore the functor $X \mapsto \{\sigma^{-1}R\}^L \otimes_R X : \mathcal{B} \rightarrow \mathcal{D}$ kills all the objects in σ . Since derived tensor product preserves triangles and coproducts, the subcategory of \mathcal{B} annihilated by $X \mapsto \{\sigma^{-1}R\}^L \otimes_R X$ must be closed under triangles and coproducts, and therefore contains all of $\mathcal{A} = D(R, \sigma)$. By the universal property of the Verdier quotient $\mathcal{C} = \mathcal{B}/\mathcal{A}$, there is a unique factorization

$$\mathcal{B} \xrightarrow{\pi} \mathcal{C} \xrightarrow{T} \mathcal{D} .$$

It remains to show that T takes $\mathcal{C}^c \subset \mathcal{C}$ to $\mathcal{D}^c \subset \mathcal{D}$.

It is clear that the map $T\pi : \mathcal{B} \rightarrow \mathcal{D}$ takes a bounded complex of f.g. projective R -modules to a bounded complex of f.g. projective $\sigma^{-1}R$ -modules; the map just tensors with $\sigma^{-1}R$. In other words, the functor $T\pi$ obviously takes \mathcal{B}^c to $\mathcal{D}^c \subset \mathcal{D}$. By 3.9.4, every object in \mathcal{C}^c is a direct summand of an object in the image of $\pi : \mathcal{B}^c \rightarrow \mathcal{C}^c$. Therefore the functor T takes any object in \mathcal{C}^c to a direct summand of an object in \mathcal{D}^c . But by Proposition 3.4 of [5], any direct summand of an object in \mathcal{D}^c lies in \mathcal{D}^c . \square

Proposition 5.2. *For any two projective R -modules P and Q , one has*

$$\mathcal{C}(\pi P, \pi Q) = \mathcal{D}(T\pi P, T\pi Q) = \text{Hom}_{\sigma^{-1}R}(\sigma^{-1}P, \sigma^{-1}Q) .$$

Proof. The identity $\mathcal{D}(T\pi P, T\pi Q) = \text{Hom}_{\sigma^{-1}R}(\sigma^{-1}P, \sigma^{-1}Q)$ is just by definition. We have defined $T\pi P = \{\sigma^{-1}R\} \times_R P = \sigma^{-1}P$.

There is a natural map, induced by the functor T ,

$$\mathcal{C}(\pi P, \pi Q) \longrightarrow \mathcal{D}(T\pi P, T\pi Q) .$$

We need to prove it an isomorphism. The case where $P = Q = R$ is easy; we have

$$\begin{aligned}
\mathcal{C}(\pi R, \pi R) &= \mathcal{B}(R, G\pi R) && \text{by adjunction} \\
&= H^0(G\pi Q) \\
&= \sigma^{-1}R && \text{by Theorem 3.21} \\
&= \mathcal{D}(T\pi R, T\pi R) .
\end{aligned}$$

But the collection of all P and Q for which the map $T : \mathcal{C}(\pi P, \pi Q) \longrightarrow \mathcal{D}(T\pi P, T\pi Q)$ is an isomorphism is clearly closed under direct sums and direct summands, and hence contains all projective modules. \square

In Section 8, we shall need to know that the only object of \mathcal{C}^c annihilated by T is the zero object. The next two propositions prove this.

Proposition 5.3. *For any object X in $\mathcal{B}^c = D^c(R)$, we have the implication*

$$\{\{\sigma^{-1}R\}^L \otimes_R X = 0\} \implies \{X \in \mathcal{A}^c\}.$$

Proof. Take any $X \in \mathcal{B}^c$. Since $X \in \mathcal{B}^c = D^c(R)$, we know that X is isomorphic to a bounded complex of f.g. projective R -modules. Up to suspension, X may be written as a complex

$$\longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0 \longrightarrow$$

If $\{\sigma^{-1}R\}^L \otimes_R X = 0$, then the complex

$$\longrightarrow 0 \longrightarrow \sigma^{-1}X^0 \longrightarrow \cdots \longrightarrow \sigma^{-1}X^n \longrightarrow 0 \longrightarrow$$

must be contractible. There are maps $\sigma^{-1}X^i \longrightarrow \sigma^{-1}X^{i-1}$ so that, for each i , the sum of the two composites

$$\begin{array}{ccc}
\sigma^{-1}X^i & \longrightarrow & \sigma^{-1}X^{i+1} \\
\downarrow & & \downarrow \\
\sigma^{-1}X^{i-1} & \longrightarrow & \sigma^{-1}X^i
\end{array}$$

is the identity on $\sigma^{-1}X^i$. By Proposition 5.2, the contracting homotopy may be lifted to the complex

$$\longrightarrow 0 \longrightarrow \pi X^0 \xrightarrow{\partial} \pi X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \pi X^{n-1} \xrightarrow{\partial} \pi X^n \longrightarrow 0 \longrightarrow$$

For each i there are maps $\pi X^i \longrightarrow \pi X^{i-1}$, so that the two composites

$$\begin{array}{ccc}
\pi X^i & \xrightarrow{\partial} & \pi X^{i+1} \\
D \downarrow & & \downarrow D \\
\pi X^{i-1} & \xrightarrow{\partial} & \pi X^i
\end{array}$$

add to the identity on πX^i .

Now let $Y^i \in \mathcal{B}^c$ be the complex

$$\longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^{i-1} \longrightarrow X^i \longrightarrow 0 \longrightarrow$$

For each i , there is a triangle

$$\Sigma^{-i-1}X^{i+1} \longrightarrow Y^{i+1} \longrightarrow Y^i \longrightarrow \Sigma^{-i}X^{i+1}.$$

The functor π is triangulated, and hence for each i we deduce a triangle

$$\Sigma^{-i-1}\pi X^{i+1} \longrightarrow \pi Y^{i+1} \longrightarrow \pi Y^i \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1}.$$

We shall prove, by induction on i , that

(i) The map

$$\pi Y^i \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1}$$

is a split monomorphism in $\mathcal{B} = D(R)$.

(ii) For each i we shall produce an explicit splitting; that is, we shall produce a map

$$\Sigma^{-i}\pi X^{i+1} \xrightarrow{\theta_i} \pi Y^i$$

so that $\theta_i \rho_i$ is the identity on πY^i .

(iii) $1 - \rho_i \theta_i$ is an endomorphism of $\Sigma^{-i}\pi X^{i+1}$. We shall show it to be the composite

$$\Sigma^{-i}\pi X^{i+1} \xrightarrow{\Sigma \partial} \Sigma^{-i}\pi X^{i+2} \xrightarrow{\Sigma D} \Sigma^{-i}\pi X^{i+1}$$

with ∂ and D as above, satisfying $1 = D\partial + \partial D$.

Note that for $i < -1$, $X^{i+1} = Y^i = 0$, and there is nothing to do. We may assume that (i)-(iii) hold for some i . We only need to show the induction step; that is, if it holds for i then it holds also for $i+1$.

It is easy to compute, in the derived category $D(R)$, the composite $\alpha\beta$, with α and β the morphisms in the triangles below

$$\begin{aligned} \Sigma^{-i-1}X^{i+1} &\xrightarrow{\beta} Y^{i+1} \longrightarrow Y^i \longrightarrow \Sigma^{-i}X^{i+1} \\ \Sigma^{-i-2}X^{i+2} &\longrightarrow Y^{i+2} \longrightarrow Y^{i+1} \xrightarrow{\alpha} \Sigma^{-i-1}X^{i+2} \end{aligned}$$

The morphism $\alpha\beta$ is just Σ^{-i-1} applied to the differential $X^{i+1} \longrightarrow X^{i+2}$. Applying the functor π we conclude the following. By the part (iii) of the induction hypothesis, the composite

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\partial} \Sigma^{-i-1}\pi X^{i+2} \xrightarrow{D} \Sigma^{-i-1}\pi X^{i+1}$$

is equal to $1 - \Sigma^{-1}(\rho_i \theta_i)$. By the above, it factors further as

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \xrightarrow{\pi\alpha} \Sigma^{-i-1}\pi X^{i+2} \xrightarrow{D} \Sigma^{-i-1}\pi X^{i+1}$$

Now look at the longer composite

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \xrightarrow{D\circ(\pi\alpha)} \Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1}$$

It is equal to $(\pi\beta)[1 - \Sigma^{-1}(\rho_i \theta_i)]$. The distinguished triangle

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \longrightarrow \pi Y^i \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1}$$

coupled with the fact that ρ_i is a split monomorphism, guarantees that the triangle is really a split exact sequence in \mathcal{C}

$$0 \longrightarrow \Sigma^{-1}\pi Y^i \xrightarrow{\Sigma^{-1}\rho_i} \Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi \Sigma Y^{i+1} \longrightarrow 0.$$

But then $\pi\beta$ is a split epimorphism, and its composite with $\Sigma^{-1}\rho_i$ vanishes. From the vanishing of $\{\pi\beta\}\{\Sigma^{-1}\rho\}$ it follows that

$$(\pi\beta)[1 - \Sigma^{-1}(\rho_i\theta_i)] = \pi\beta,$$

and hence that

$$[1 - (\pi\beta) \circ D \circ (\pi\alpha)](\pi\beta) = 0.$$

Since $\pi\beta$ is a split epimorphism, we conclude that

$$(\pi\beta) \circ D \circ (\pi\alpha) = 1.$$

But $\pi\alpha : \pi Y^{i+1} \longrightarrow \Sigma^{-i-1}\pi X^{i+2}$ is nothing other than the map ρ_{i+1} , and if we put $\theta_{i+1} = (\pi\beta) \circ D$, then we have proved parts (i) and (ii) for $i+1$.

It only remains to establish (iii). But by construction, $\Sigma\rho_{i+1}\Sigma\theta_{i+1}$ is given by the composite

$$\Sigma^{-i}\pi X^{i+2} \xrightarrow{D} \Sigma^{-i}\pi X^{i+1} \xrightarrow{\pi\alpha} \pi\Sigma Y^{i+1} \xrightarrow{\Sigma\rho_{i+1}} \Sigma^{-i}\pi X^{i+2},$$

which is nothing other than a suspension of ∂D . Hence this equals $1 - D\partial$.

This completes the induction. Now choose $i > n$. The complex Y^i is nothing other than $X \in \mathcal{B}$, and by (i) we conclude that πX is a direct summand of $\pi X^{i+1} = 0$. It follows that $\pi X = 0$. This forces $X \in \mathcal{A}$, but we know that $X \in \mathcal{B}^c$. Hence $X \in \mathcal{A} \cap \mathcal{B}^c = \mathcal{A}^c$. \square

Proposition 5.4. *Suppose x is an object in \mathcal{C}^c , and suppose $Tx = 0$, where*

$$T\pi : \mathcal{B} = D(R) \longrightarrow \mathcal{D} = D(\sigma^{-1}R) ; x \mapsto \sigma^{-1}x$$

is the functor induced by tensor with $\sigma^{-1}R$, as in Proposition 5.1. Then $x = 0$.

Proof. By Proposition 5.3 we know that if $x \in \mathcal{B}^c$, and if

$$T\pi x = \{\sigma^{-1}R\}^L \otimes_R x = 0,$$

then $x \in \mathcal{A}^c$, in other words $\pi x = 0$. The Proposition is therefore true for all objects $\pi x \in \mathcal{C}^c$, with $x \in \mathcal{B}^c$.

By 3.9.4, the map $\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c$ is fully faithful, and \mathcal{C}^c is the smallest thick subcategory containing $\mathcal{B}^c/\mathcal{A}^c \subset \mathcal{C}^c$. By Corollary 4.5.12 of [20] we conclude the following. For any object $t \in \mathcal{C}^c$ there exists an object $x \in \mathcal{B}^c$ with $t \oplus \Sigma t \simeq \pi x$. Then $0 = Tt \oplus \Sigma Tt \simeq T\pi x$, and by the above this means $\pi x = 0$. But t is a direct summand of πx ; hence $t = 0$. \square

6. CHAIN COMPLEX LIFTING

We consider the chain complex lifting problem of deciding if a bounded complex D of f.g. projective $\sigma^{-1}R$ -modules is chain equivalent to $\sigma^{-1}C$ for a bounded complex C of f.g. projective R -modules. In terms of the functor

$$T\pi : \mathcal{B} = D(R) \longrightarrow \mathcal{D} = D(\sigma^{-1}R) ; X \mapsto \{\sigma^{-1}R\}^L \otimes_R X$$

the problem is to decide if a compact object D in \mathcal{D} lifts to a compact object C in \mathcal{B} .

To have any chance, we must start with a complex of induced f.g. projective $\sigma^{-1}R$ -modules, of the form

$$D : \dots \longrightarrow \sigma^{-1}x^{i-1} \longrightarrow \sigma^{-1}x^i \longrightarrow \sigma^{-1}x^{i+1} \longrightarrow \dots$$

with the x^i 's f.g. projective R -modules. We can write D as

$$D : \dots \longrightarrow T\pi x^{i-1} \longrightarrow T\pi x^i \longrightarrow T\pi x^{i+1} \longrightarrow \dots$$

By Proposition 5.2, this may be lifted uniquely to a chain complex of objects in \mathcal{C}^c

$$\tilde{D} : \dots \longrightarrow \pi x^{i-1} \longrightarrow \pi x^i \longrightarrow \pi x^{i+1} \longrightarrow \dots$$

with $\mathcal{C} = \mathcal{B}/\mathcal{A} = D(R)/D(R, \sigma)$. Because all the objects lie in the image of $\pi : \mathcal{B}^c \longrightarrow \mathcal{C}^c$ and because the functor $\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c$ is fully faithful (see 3.9.4), we may view the chain complex \tilde{D} , uniquely, as lying in $\mathcal{B}^c/\mathcal{A}^c$. The next results discuss lifting this to \mathcal{B}^c .

Lemma 6.1. *Given any diagram in $\mathcal{B}^c/\mathcal{A}^c$ of the form*

$$\begin{array}{ccc} \pi x^0 & \longrightarrow & \pi x^1 \\ \downarrow \wr & & \\ \pi y^0 & & \end{array}$$

where the vertical map is an isomorphism, we may complete to a commutative square

$$\begin{array}{ccc} \pi x^0 & \longrightarrow & \pi x^1 \\ \downarrow \wr & & \downarrow \wr \\ \pi y^0 & \xrightarrow{\pi f} & \pi y^1 \end{array}$$

where both vertical maps are isomorphisms in $\mathcal{B}^c/\mathcal{A}^c$, and $\pi f : \pi y^0 \longrightarrow \pi y^1$ is obtained by applying the functor π to some map $f : y^0 \longrightarrow y^1$.

Proof. In $\mathcal{B}^c/\mathcal{A}^c$, we have a map $\pi y^0 \longrightarrow \pi x^1$. Such maps are equivalence classes of diagrams in \mathcal{B}^c

$$\begin{array}{ccc} & x^1 & \\ & \downarrow \alpha & \\ y^0 & \xrightarrow{f} & y^1 \end{array}$$

with $\pi\alpha$ an isomorphism. Taking π of this, we get our commutative square. \square

After these preliminaries, we return to the problem of lifting a bounded chain complex of induced f.g. projective $\sigma^{-1}R$ modules

$$D : \dots \longrightarrow \sigma^{-1}x^{i-1} \longrightarrow \sigma^{-1}x^i \longrightarrow \sigma^{-1}x^{i+1} \longrightarrow \dots$$

to a bounded chain complex of f.g. projective R -modules C with a chain equivalence $\sigma^{-1}C \simeq D$. We shall only treat the special case of a complex of length 3 in detail, but the general case is similar to this one.

We begin with

$$D : \longrightarrow 0 \longrightarrow \sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^1 \longrightarrow \sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3 \longrightarrow 0 \longrightarrow$$

In the derived category $\mathcal{D}^c = D^c(\sigma^{-1}R)$, we have several distinguished triangles. We wish to consider three of them. They are given by the mapping cones

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & 0 & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^2 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & \sigma^{-1}x^2 & \longrightarrow & 0 & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & \sigma^{-1}x^2 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^3 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & \sigma^{-1}x^0 & \longrightarrow & \sigma^{-1}x^1 & \longrightarrow & \sigma^{-1}x^2 & \longrightarrow & \sigma^{-1}x^3 & \longrightarrow & 0 & \longrightarrow \end{array}$$

The more abstract way of stating this is as follows. In the derived category $D^c(\sigma^{-1}R)$, the map $\sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^1$ may be completed to a triangle

$$\sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^1 \longrightarrow X_1 \longrightarrow \Sigma\sigma^{-1}x^0 .$$

This is the first of our three distinguished triangles above. Because the composite $\sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^1 \longrightarrow \sigma^{-1}x^2$ vanishes, we may factor the map $\sigma^{-1}x^1 \longrightarrow \sigma^{-1}x^2$ as $\sigma^{-1}x^1 \longrightarrow X_1 \longrightarrow \sigma^{-1}x^2$. The factorization $X_1 \longrightarrow \sigma^{-1}x^2$ is unique, since its ambiguity is up to a map $\Sigma\sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^2$, which must vanish because it is a map from an object in $D(\sigma^{-1}R)^{\leq -1}$ to an object in $D(\sigma^{-1}R)^{\geq 0}$. The composite $X_1 \longrightarrow \sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3$ must be zero, because it is the unique factorization of the zero map $\sigma^{-1}x^1 \longrightarrow \sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3$ through $\sigma^{-1}x^1 \longrightarrow X_1$. Next complete $X_1 \longrightarrow \sigma^{-1}x^2$ to a triangle

$$X_1 \longrightarrow \sigma^{-1}x^2 \longrightarrow X_2 \longrightarrow \Sigma X_1 .$$

This is the second of our three triangles above. The vanishing of the composite $X_1 \longrightarrow \sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3$ tells us that the map $\sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3$ must factor as $\sigma^{-1}x^2 \longrightarrow X_2 \longrightarrow \sigma^{-1}x^3$. The factorization is unique up to a morphism $\Sigma^{-1}X_1 \longrightarrow \sigma^{-1}x^3$, and all such maps vanish because $\Sigma^{-1}X_1 \in D(\sigma^{-1}R)^{\leq -1}$ while $\sigma^{-1}x^3 \in D(\sigma^{-1}R)^{\geq 0}$. Finally, we may complete $X_2 \longrightarrow \sigma^{-1}x^3$ to a triangle

$$X_2 \longrightarrow \sigma^{-1}x^3 \longrightarrow X_3 \longrightarrow \Sigma X_2 .$$

This gives the third triangle above. The question is whether this construction can be lifted to $D^c(R)$. Since in $D^c(\sigma^{-1}R)$ the choices of factorizations were all unique, any lifting of the triangles and factorizations to $D^c(R)$ will map, under the functor $T\pi$, to the above. The question is only whether the diagram of distinguished triangles just constructed exists in $D^c(R)$. We treat first the problem of lifting by the functor π . The obstructions are the well-known Toda brackets (= Massey products) – see Chapter IV.3 of Gelfand and Manin [12]. We give a detailed treatment of this only in the simplest case, of a 3-dimensional complex.

Theorem 6.2. *Let D be a 3-dimensional chain complex of induced f.g. projective $\sigma^{-1}R$ -modules*

$$D : \sigma^{-1}x^0 \longrightarrow \sigma^{-1}x^1 \longrightarrow \sigma^{-1}x^2 \longrightarrow \sigma^{-1}x^3 ,$$

which we rewrite as

$$D : T\pi x^0 \longrightarrow T\pi x^1 \longrightarrow T\pi x^2 \longrightarrow T\pi x^3 .$$

By Proposition 5.2, D may be lifted uniquely to a chain complex in \mathcal{C}^c

$$\tilde{D} : \pi x^0 \xrightarrow{f} \pi x^1 \xrightarrow{g} \pi x^2 \xrightarrow{h} \pi x^3 .$$

There is defined an element

$$\theta(D) \in \frac{\mathcal{C}^c(\Sigma\pi x^0, \pi x^3)}{\text{Im}\{\mathcal{C}^c(\Sigma\pi x^0, \pi x^2) \oplus \mathcal{C}^c(\Sigma\pi x^1, \pi x^3)\}}$$

such that the following conditions are equivalent :

- (i) $\theta(D) = 0$.
- (ii) *There exist three triangles in \mathcal{C}^c*

$$\begin{array}{ccccccc} \pi x^0 & \xrightarrow{f} & \pi x^1 & \xrightarrow{\alpha} & X_1 & \longrightarrow & \Sigma\pi x^0 \\ X_1 & \xrightarrow{\beta} & \pi x^2 & \xrightarrow{\gamma} & X_2 & \longrightarrow & \Sigma X_1 \\ X_2 & \xrightarrow{\delta} & \pi x^3 & \longrightarrow & X_3 & \longrightarrow & \Sigma X_2 \end{array}$$

such that $g = \beta\alpha$ and $h = \delta\gamma$.

(iii) *There exist three triangles in $\mathcal{B}^c = D^c(R)$*

$$\begin{array}{ccccccc} y^0 & \xrightarrow{\tilde{f}} & y^1 & \xrightarrow{\alpha} & Y_1 & \longrightarrow & \Sigma y^0 \\ Y_1 & \xrightarrow{\beta} & y^2 & \xrightarrow{\gamma} & Y_2 & \longrightarrow & \Sigma Y_1 \\ Y_2 & \xrightarrow{\delta} & y^3 & \longrightarrow & Y_3 & \longrightarrow & \Sigma Y_2 \end{array}$$

and an isomorphism of chain complexes in \mathcal{C}^c

$$\begin{array}{ccccccc} \pi x^0 & \xrightarrow{f} & \pi x^1 & \xrightarrow{g} & \pi x^2 & \xrightarrow{h} & \pi x^3 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \pi y^0 & \xrightarrow{\pi \tilde{f}} & \pi y^1 & \xrightarrow{\pi(\beta\alpha)} & \pi y^2 & \xrightarrow{\pi(\delta\gamma)} & \pi y^3 \end{array}$$

In particular, Y_3 is a bounded f.g. projective R -module chain complex such that $\sigma^{-1}Y_3 \simeq D$, solving the lifting problem.

Proof. In the first instance, we define $\theta(D)$. We may always complete f to a triangle

$$\pi x^0 \xrightarrow{f} \pi x^1 \xrightarrow{\alpha} X_1 \longrightarrow \Sigma \pi x^0 .$$

The fact that $gf = 0$ permits us to factor g as

$$\pi x^1 \xrightarrow{\alpha} X_1 \xrightarrow{\beta} \pi x^2 .$$

But β is not well-defined; we may change our choice by any element $\phi \in \mathcal{C}^c(\Sigma \pi x^0, \pi x^2)$.

Now we may study the maps

$$\pi x^1 \xrightarrow{\alpha} X_1 \xrightarrow{\beta} \pi x^2 \xrightarrow{h} \pi x^3 .$$

The composite $h\beta\alpha = hg = 0$. We cannot be certain that $h\beta$ vanishes, but we know that $h\beta$ composes with α to give zero. From the triangle above, it follows that $h\beta$ factors through

$$X_1 \longrightarrow \Sigma \pi x^0 \xrightarrow{\theta} \pi x^3 .$$

The composite $h\beta$ will vanish if and only if the map θ factors further as

$$\Sigma \pi x^0 \xrightarrow{\Sigma f} \Sigma \pi x^1 \xrightarrow{\psi} \pi x^3 .$$

We deduce that there is an obstruction to continuing the process, given by $\theta \in \mathcal{C}^c(\Sigma \pi x^0, \pi x^3)$.

And this θ is well defined up to adding a $\phi \in \mathcal{C}^c(\Sigma \pi x^0, \pi x^2)$ and a $\psi \in \mathcal{C}^c(\Sigma \pi x^1, \pi x^3)$.

The element defined by

$$\theta(D) = [\theta] \in \frac{\mathcal{C}^c(\Sigma \pi x^0, \pi x^3)}{\text{Im}\{\mathcal{C}^c(\Sigma \pi x^0, \pi x^2) \oplus \mathcal{C}^c(\Sigma \pi x^1, \pi x^3)\}}$$

is such that $\theta(D) = 0$ if and only if we may choose β so that $h\beta = 0$.

We now prove (i) \iff (ii) \iff (iii).

(iii) \implies (ii) \implies (i) Obvious.

(i) \implies (ii) As above, factor g as $\beta\alpha$ so that $h\beta = 0$. Complete β to a triangle

$$X_1 \xrightarrow{\beta} \pi x^2 \xrightarrow{\gamma} X_2 \longrightarrow \Sigma X_1 .$$

Because $h\beta = 0$, we may factor h as

$$\pi x^2 \xrightarrow{\gamma} X_2 \xrightarrow{\delta} \pi x^3 .$$

Complete δ to a triangle

$$X_2 \xrightarrow{\delta} \pi x^3 \longrightarrow X_3 \longrightarrow \Sigma X_2,$$

and we are done.

(ii) \implies (iii) We may assume that we are given three triangles in $\mathcal{B}^c/\mathcal{A}^c$

$$\begin{array}{ccccccc} \pi x^0 & \xrightarrow{f} & \pi x^1 & \xrightarrow{\alpha'} & X_1 & \longrightarrow & \Sigma \pi x^0 \\ X_1 & \xrightarrow{\beta'} & \pi x^2 & \xrightarrow{\gamma'} & X_2 & \longrightarrow & \Sigma X_1 \\ X_2 & \xrightarrow{\delta'} & \pi x^3 & \longrightarrow & X_3 & \longrightarrow & \Sigma X_2 \end{array}$$

such that $g = \beta'\alpha'$ and $h = \delta'\gamma'$. Put $y^0 = x^0$. Applying Lemma 6.1 to the diagram

$$\begin{array}{ccc} \pi x^0 & \xrightarrow{f} & \pi x^1 \\ 1 \downarrow & & \\ \pi y^0 & & \end{array}$$

we may complete to a commutative square

$$\begin{array}{ccc} \pi x^0 & \xrightarrow{f} & \pi x^1 \\ 1 \downarrow & & \downarrow \wr \\ \pi y^0 & \xrightarrow{\pi \tilde{f}} & \pi y^1 . \end{array}$$

Form the triangle

$$y^0 \xrightarrow{\tilde{f}} y^1 \xrightarrow{\alpha} Y_1 \longrightarrow \Sigma y^0 .$$

Applying the functor π , we have a commutative diagram

$$\begin{array}{ccccccc} \pi x^0 & \xrightarrow{f} & \pi x^1 & \xrightarrow{\alpha'} & X_1 & \longrightarrow & \Sigma \pi x^0 \\ 1 \downarrow & & \downarrow \wr & & & & \downarrow \wr \\ \pi y^0 & \xrightarrow{\pi \tilde{f}} & \pi y^1 & \xrightarrow{\pi \alpha} & \pi Y_1 & \longrightarrow & \Sigma \pi y^0 \end{array}$$

which may be extended to an isomorphism of triangles. We have a commutative diagram

$$\begin{array}{ccccc} \pi x^1 & \xrightarrow{\alpha'} & X_1 & \xrightarrow{\beta'} & \Sigma \pi x^2 \\ \downarrow \wr & & \downarrow \wr & & \\ \pi y^1 & \xrightarrow{\pi \alpha} & \pi Y_1 & & \end{array}$$

which, again by Lemma 6.1, we may extend to

$$\begin{array}{ccccc} \pi x^1 & \xrightarrow{\alpha'} & X_1 & \xrightarrow{\beta'} & \pi x^2 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \pi y^1 & \xrightarrow{\pi\alpha} & \pi Y_1 & \xrightarrow{\pi\beta} & \pi y^2 . \end{array}$$

Now complete β to a triangle

$$Y_1 \xrightarrow{\beta} y^2 \xrightarrow{\gamma} Y_2 \longrightarrow \Sigma Y_1 .$$

Again, we have a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\beta'} & \pi x^2 & \xrightarrow{\gamma'} & X_2 & \longrightarrow & \Sigma X_1 \\ 1 \downarrow & & \downarrow \wr & & & & \downarrow \wr \\ \pi Y_1 & \xrightarrow{\pi\beta} & \pi y^2 & \xrightarrow{\pi\gamma} & \pi Y_2 & \longrightarrow & \Sigma \pi Y_1 \end{array}$$

which we extend to an isomorphism of triangles. Lemma 6.1 allows us to extend the commutative diagram

$$\begin{array}{ccccc} \pi x^2 & \xrightarrow{\gamma'} & X_2 & \xrightarrow{\delta'} & \pi x^3 \\ 1 \downarrow & & \downarrow \wr & & \\ \pi y^2 & \xrightarrow{\pi\gamma} & \pi Y_2 & & \end{array}$$

to the diagram

$$\begin{array}{ccccc} \pi x^2 & \xrightarrow{\gamma'} & X_2 & \xrightarrow{\delta'} & \pi x^3 \\ 1 \downarrow & & \downarrow \wr & & \downarrow \wr \\ \pi y^2 & \xrightarrow{\pi\gamma} & \pi Y_2 & \xrightarrow{\pi\delta} & \pi y^3 \end{array}$$

Finally, we form the triangle

$$Y_2 \xrightarrow{\delta} y^3 \longrightarrow Y_3 \longrightarrow \Sigma Y_2$$

□

Lemma 6.3. *Let M and N be f.g. projective R -modules. There is a natural isomorphism*

$$\mathcal{C}^c(\Sigma\pi M, \pi N) \cong \mathrm{Tor}_2^R(\sigma^{-1}M^*, \sigma^{-1}N).$$

In this formula, $M^ = \mathrm{Hom}_R(M, R)$ is the dual of M .*

Proof. By Corollary 3.27, $\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0$. The case $n = 1$ of Corollary 3.31 then tell us that

$$\begin{aligned} \mathrm{Tor}_2^R(\sigma^{-1}R, \sigma^{-1}R) &= H^{-1}(G\pi R) \\ &= \mathcal{B}(\Sigma R, G\pi R) \\ &= \mathcal{C}^c(\Sigma\pi R, \pi R). \end{aligned}$$

All we are doing is extending this isomorphism first to free modules, then to their direct summands. \square

Remark 6.4. Lemma 6.3 permits us to write the obstruction class $\theta(D)$ of Theorem 6.2 as lying in the group

$$\frac{\mathrm{Tor}_2^R(\sigma^{-1}\{x^0\}^*, \sigma^{-1}x^3)}{\mathrm{Im}\{\mathrm{Tor}_2^R(\sigma^{-1}\{x^0\}^*, \sigma^{-1}x^2) \oplus \mathrm{Tor}_2^R(\sigma^{-1}\{x^1\}^*, \sigma^{-1}x^0)\}}.$$

Note that if $\mathrm{Tor}_2^R(\sigma^{-1}R, \sigma^{-1}R) = 0$ this group is 0. \square

Remark 6.5. It is easy to generalize this to longer complexes. Given a bounded f.g. projective R -module chain complex

$$D : \cdots \longrightarrow \sigma^{-1}x^{i-1} \longrightarrow \sigma^{-1}x^i \longrightarrow \sigma^{-1}x^{i+1} \longrightarrow \cdots$$

there is a series of obstructions to lifting all the associated triangles to $D^c(R)$

$$\theta_{i,j}(D) \in \frac{\mathcal{C}^c(\Sigma^{j-2}\pi x^i, \pi x^{i+j})}{\mathrm{Im}\{\mathcal{C}^c(\Sigma^{j-2}\pi x^i, \pi x^{i+j-1}) \oplus \mathcal{C}^c(\Sigma^{j-2}\pi x^{i+1}, \pi x^{i+j})\}}$$

for $j \geq 3$. As in Remark 6.4 these are related to Tor-groups by a spectral sequence, and are 0 if $\mathrm{Tor}_*^R(\sigma^{-1}R, \sigma^{-1}R) = 0$ for $* \geq 2$. \square

7. WALDHAUSEN'S APPROXIMATION AND LOCALIZATION THEOREMS

We have been studying noncommutative localization using derived categories techniques. Next we want to apply our results to deduce K -theoretic consequences. In order to do so, we briefly review some results of Waldhausen's.

Let \mathbf{C} be a category with cofibrations and weak equivalences. Out of \mathbf{C} Waldhausen constructs a spectrum, denoted $K(\mathbf{C})$. In Thomason's [29], the category \mathbf{C} is assumed to be a full subcategory of the category of chain complexes over some abelian category, the cofibrations are maps of complexes which are split monomorphisms in each degree, and the weak equivalences are the quasi-isomorphisms. We shall call such categories *permissible Waldhausen categories*.

Remark 7.1. Thomason's term for them is *complicial biWaldhausen categories*. \square

Given a permissible Waldhausen category \mathbf{C} , one can form its derived category; just invert the weak equivalences. We denote this derived category by $D(\mathbf{C})$. We have two major theorems here, both of which are special cases of more general theorems of Waldhausen.

Theorem 7.2. (Waldhausen's Approximation Theorem). *Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be an exact functor of essentially small permissible Waldhausen categories (categories of chain complexes, as above). Suppose that the induced map of derived categories*

$$D(F) : D(\mathbf{C}) \longrightarrow D(\mathbf{D})$$

is an equivalence of categories. Then the induced map of spectra

$$K(F) : K(\mathbf{C}) \longrightarrow K(\mathbf{D})$$

is a homotopy equivalence.

In this sense, Waldhausen's K -theory is almost an invariant of the derived categories. To construct it, one needs to have a great deal more structure. One must begin with a permissible category with cofibrations and weak equivalences. But the Approximation Theorem asserts that the dependence on the added structure is not strong.

Theorem 7.3. (Waldhausen's Localization Theorem). *Let \mathbf{A} , \mathbf{B} and \mathbf{C} be essentially small permissible Waldhausen categories. Suppose*

$$\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$$

are exact functors of permissible Waldhausen categories. Suppose further that the induced triangulated functors of derived categories

$$D(\mathbf{A}) \longrightarrow D(\mathbf{B}) \longrightarrow D(\mathbf{C})$$

compose to zero, and that the natural map

$$D(\mathbf{B})/D(\mathbf{A}) \longrightarrow D(\mathbf{C})$$

is an equivalence of categories. Then the sequence of spectra

$$K(\mathbf{A}) \longrightarrow K(\mathbf{B}) \longrightarrow K(\mathbf{C})$$

is a homotopy fibration. □

To obtain a homotopy fibration using Waldhausen's localization theorem, we need to produce three permissible Waldhausen categories, and a sequence

$$\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$$

so that

$$D(\mathbf{B})/D(\mathbf{A}) \longrightarrow D(\mathbf{C})$$

is an equivalence of categories. In particular, we want to find triangulated categories $\mathcal{A}^c = D(\mathbf{A})$, $\mathcal{B}^c = D(\mathbf{B})$ and $\mathcal{C}^c = D(\mathbf{C})$ so that $\mathcal{C}^c = \mathcal{B}^c/\mathcal{A}^c$. Of course, it is not enough to just find the triangulated categories \mathcal{A}^c , \mathcal{B}^c and \mathcal{C}^c ; to apply the localization theorem, we must also find the permissible Waldhausen categories \mathbf{A} , \mathbf{B} and \mathbf{C} , and the exact functors

$$\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C} .$$

In Theorem 7.2, we learned that the K -theory is largely independent of the choices of \mathbf{A} , \mathbf{B} and \mathbf{C} . In this article, we shall allow ourselves some latitude. Thomason is careful to check, in [29], that the choices of permissible Waldhausen categories can be made; we shall consider this a technical point, and explain only how to produce $\mathcal{A}^c = D(\mathbf{A})$, $\mathcal{B}^c = D(\mathbf{B})$ and $\mathcal{C}^c = D(\mathbf{C})$. We shall also commit the notational sin of writing $K(\mathcal{A}^c)$ for $K(\mathbf{A})$, where $\mathcal{A}^c = D(\mathbf{A})$, and similarly $K(\mathcal{B}^c)$ for $K(\mathbf{B})$, and $K(\mathcal{C}^c)$ for $K(\mathbf{C})$.

As the notation of the previous paragraph was designed to suggest, we want to apply the results $\mathcal{A}^c \subset \mathcal{B}^c$ and \mathcal{C}^c , with \mathcal{A} , \mathcal{B} and \mathcal{C} as we have seen them in the previous sections. That is, $\mathcal{B} = D(R)$ is the derived category of a ring R , \mathcal{A} is generated by a set σ of morphisms in \mathcal{B}^c , and $\mathcal{C} = \mathcal{B}/\mathcal{A}$. By the discussion above, we have a fibration in K -theory

$$K(\mathcal{A}^c) \longrightarrow K(\mathcal{B}^c) \longrightarrow K(\mathcal{B}^c/\mathcal{A}^c) .$$

In Theorem 3.9 we learned that the natural map $\mathcal{B}^c/\mathcal{A}^c \rightarrow \mathcal{C}^c$ is fully faithful, and that up to splitting idempotents it is an equivalence. Grayson's cofinality theorem then tells us that the map

$$K(\mathcal{B}^c/\mathcal{A}^c) \longrightarrow K(\mathcal{C}^c)$$

induces an isomorphism $K_i(\mathcal{B}^c/\mathcal{A}^c) \rightarrow K_i(\mathcal{C}^c)$ when $i > 0$, while $K_0(\mathcal{B}^c/\mathcal{A}^c) \rightarrow K_0(\mathcal{C}^c)$ is injective. We conclude that, up to the failure of surjectivity in π_0 ,

$$K(\mathcal{A}^c) \longrightarrow K(\mathcal{B}^c) \longrightarrow K(\mathcal{C}^c)$$

is a homotopy fibration.

We know also that $\mathcal{B}^c = D^c(R)$, and in Proposition 5.2 we produced a functor $T : \mathcal{C}^c \rightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$. For any ring S , we have $K(S) = K(D^c(S))$; Waldhausen's K -theory of the derived category agrees with Quillen's K -theory of S . Applying this to the rings R and $\sigma^{-1}R$, we have

Theorem 7.4. *In the diagram*

$$\begin{array}{ccccc} K(\mathcal{A}^c) & \longrightarrow & K(\mathcal{B}^c) & \longrightarrow & K(\mathcal{C}^c) \\ & & \downarrow \wr & & \downarrow K(T) \\ & & K(R) & & K(\sigma^{-1}R) \end{array}$$

not only is the top row a fibration up to the failure of surjectivity on π_0 , but $K(\mathcal{B}^c)$ agrees with Quillen's $K(R)$, and $T : \mathcal{C}^c \rightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$ induces a morphism $K(T) : K(\mathcal{C}^c) \rightarrow K(\sigma^{-1}R)$. \square

In particular, Theorem 7.4 gives a long exact sequence

$$\dots \rightarrow K_n(R) \rightarrow K_n(\mathcal{C}^c) \rightarrow K_n(R, \sigma) \rightarrow K_{n-1}(R) \rightarrow \dots ,$$

with $K_*(R, \sigma) = K_{*-1}(\mathcal{A}^c)$. In the coming sections, we shall study the range in which the map $K(T) : K_*(\mathcal{C}^c) \rightarrow K_*(\sigma^{-1}R)$ induces an isomorphism in homotopy.

8. T INDUCES A K_0 -ISOMORPHISM

Let the notation be as in Theorem 7.4. In this section, we shall prove that the functor

$$T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$$

induces an isomorphism in K_0 . We shall do it through a sequence of lemmas. We remind the reader that $\mathcal{B} = D(R)$ has a standard t -structure, and that the functor $G\pi$ behaves well with respect to it. See Remark 3.15 and Lemma 3.17.

Lemma 8.1. *Let n be an integer. Let $X \in \mathcal{B}^c$ be an object of $\mathcal{B}^{\leq n}$, and let P be a f.g. projective R -module. Then the functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ of Proposition 5.1 gives a homomorphism*

$$\mathcal{C}^c(\pi\Sigma^{-n}P, \pi X) \longrightarrow \mathcal{D}^c(T\pi\Sigma^{-n}P, T\pi X) .$$

We assert that this map is an isomorphism.

Proof. By translation, we may assume $n = 0$. We need to prove the map injective and surjective. Let us prove surjectivity first. Recall that $X \in \mathcal{B}^{\leq 0}$ is isomorphic to a chain complex of f.g. projectives

$$\longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

This makes $T\pi X$ the chain complex

$$\longrightarrow \sigma^{-1}X^m \longrightarrow \sigma^{-1}X^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}X^{-1} \longrightarrow \sigma^{-1}X^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

Let P be a f.g. projective R -module, concentrated in degree 0. Now the complex of $\sigma^{-1}R$ -modules $T\pi P$ is a single projective module $\sigma^{-1}P$, concentrated in degree 0. Any map in the derived category, from the bounded above complex of projectives $\sigma^{-1}P = \{\sigma^{-1}R\} \otimes_R P$ to the complex $\{\sigma^{-1}R\}^L \otimes_R X$, can be represented by a chain map. There is a map $\sigma^{-1}P \longrightarrow \sigma^{-1}X_0$ inducing it. By Proposition 5.2, this comes from a map $\pi P \longrightarrow \pi X^0$. But then the composite

$$\pi P \longrightarrow \pi X^0 \longrightarrow \pi X$$

gives a map $\pi P \longrightarrow \pi X$ in \mathcal{C}^c , inducing $T\pi P \longrightarrow T\pi X$.

This proved the surjectivity. For the injectivity, note that there is a short exact sequence of chain complexes

$$\begin{array}{ccccccccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \end{array}$$

Write the corresponding triangle as

$$X^0 \longrightarrow X \longrightarrow Y \longrightarrow \Sigma X^0.$$

We have a triangle

$$\pi X^0 \longrightarrow \pi X \longrightarrow \pi Y \longrightarrow \pi \Sigma X^0.$$

Let P be a f.g. projective R -module, concentrated in degree 0. Suppose we are given a map $\pi P \longrightarrow \pi X$. Composing to Y , we deduce a map

$$\pi P \longrightarrow \pi X \longrightarrow \pi Y.$$

By adjunction, this corresponds to a map

$$P \longrightarrow G\pi Y,$$

which must vanish. After all, $Y \in \mathcal{B}^{\leq -1}$, and by Lemma 3.17 it follows that $G\pi Y$ is also in $\mathcal{B}^{\leq -1}$. The map from a projective object P in degree 0 to the complex $G\pi Y \in \mathcal{B}^{\leq -1}$ must vanish.

It follows that the map $\pi P \longrightarrow \pi X$ must factor as

$$\pi P \longrightarrow \pi X^0 \longrightarrow \pi X.$$

Now assume that the map vanishes in $\mathcal{D}^c = D^c(\sigma^{-1}R)$. That is, the composite

$$\sigma^{-1}P \longrightarrow \sigma^{-1}X^0 \longrightarrow \{\sigma^{-1}R\}^{L \otimes_R} X$$

vanishes in \mathcal{D}^c . Then it must be null homotopic. The map $\sigma^{-1}P \longrightarrow \sigma^{-1}X^0$ must factor as

$$\sigma^{-1}P \longrightarrow \sigma^{-1}X^{-1} \longrightarrow \sigma^{-1}X^0.$$

By Proposition 5.2, this tells us that the map $\pi P \longrightarrow \pi X^0$ must factor as

$$\pi P \longrightarrow \pi X^{-1} \longrightarrow \pi X^0$$

and hence the map

$$\pi P \longrightarrow \pi X^{-1} \longrightarrow \pi X^0 \longrightarrow \pi X$$

must vanish. □

Lemma 8.2. *Let n be an integer. Let Z be an object of \mathcal{C}^c , and suppose for all $r \geq n$, $H^r(TZ) = 0$. Then there is an object $X \in \mathcal{B}^c$, that is a bounded complex of projective R -modules*

$$\longrightarrow 0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \dots \longrightarrow X^{\ell-1} \longrightarrow X^\ell \longrightarrow 0 \longrightarrow$$

so that Z is a direct summand of πX , and $\ell \leq n$.

Proof. By suspending, we may assume $n = 0$. By 3.9.4, we may certainly find an X with Z a direct summand of πX . What is not clear is that we may choose X to be a complex

$$\longrightarrow 0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \dots \longrightarrow X^{\ell-1} \longrightarrow X^\ell \longrightarrow 0 \longrightarrow$$

with $\ell \leq 0$. Assume therefore that $\ell > 0$, and we shall show that we may reduce ℓ by 1.

We recall the short exact sequence of chain complexes

$$\begin{array}{ccccccccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^\ell & \longrightarrow & 0 & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{\ell-1} & \longrightarrow & X^\ell & \longrightarrow & 0 & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{\ell-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow
\end{array}$$

It gives a triangle which we write as

$$\Sigma^{-\ell} X^\ell \longrightarrow X \xrightarrow{a} Y \longrightarrow \Sigma^{-\ell+1} X^\ell.$$

We also have that Z is a direct summand of πX . That is, there are maps

$$\pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

so that $bc = 1_Z$. Now we wish to consider the composite

$$\pi \Sigma^{-\ell} X^\ell \longrightarrow \pi X \xrightarrow{b} Z \xrightarrow{c} \pi X.$$

We know that X^ℓ is a f.g. projective R -module, and $X \in \mathcal{B}$ lies in $\mathcal{B}^{\leq \ell}$. The conditions are as in Lemma 8.1. In order to prove that the composite vanishes, it suffices to prove that T of it vanishes, in $\mathcal{D}^c = D^c(\sigma^{-1}R)$.

But in \mathcal{D}^c the map becomes the composite

$$T\pi \Sigma^{-\ell} X^\ell \longrightarrow T\pi X \longrightarrow TZ \longrightarrow T\pi X.$$

We assert that already the shorter composite, $T\pi \Sigma^{-\ell} X^\ell \longrightarrow T\pi X \longrightarrow TZ$ must vanish. After all, it is a map

$$T\pi \Sigma^{-\ell} X^\ell \longrightarrow TZ$$

By hypothesis, TZ vanishes above degree 0. It is quasi-isomorphic to a complex of $\sigma^{-1}R$ -modules in degree ≤ 0 . And $T\pi \Sigma^{-\ell} X^\ell = \Sigma^{-\ell} \sigma^{-1} X^\ell$ is a single projective $\sigma^{-1}R$ -module, concentrated in degree $\ell > 0$. Hence the vanishing. The composite

$$\pi \Sigma^{-\ell} X^\ell \longrightarrow \pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

must therefore vanish. Since c is a split monomorphism, we deduce that the composite

$$\pi \Sigma^{-\ell} X^\ell \longrightarrow \pi X \xrightarrow{b} Z$$

also vanishes.

But now the triangle

$$\pi \Sigma^{-\ell} X^\ell \longrightarrow \pi X \xrightarrow{a} \pi Y \longrightarrow \pi \Sigma^{-\ell+1} X^\ell$$

tells us that the map $b : \pi X \longrightarrow Z$ must factor as

$$\pi X \xrightarrow{a} \pi Y \xrightarrow{\beta} Z.$$

The composite

$$Z \xrightarrow{c} \pi X \xrightarrow{a} \pi Y \xrightarrow{\beta} Z$$

is the identity, and hence Z is a direct summand of πY , with Y the complex

$$\longrightarrow 0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{\ell-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

□

Lemma 8.3. *Let n be an integer. Let Z be an object of \mathcal{C}^c , and suppose for all $r \geq n$, $H^r(TZ) = 0$. Given any f.g. projective R -module P , and any map*

$$T\pi P = \sigma^{-1}P \xrightarrow{a} H^n(TZ),$$

there is a map in \mathcal{C}^c

$$\pi \Sigma^{-n}P \xrightarrow{\mu} Z$$

so that $H^n(T\mu) = a$.

Proof. By translating, we may assume $n = 0$. Let Z be an object of \mathcal{C}^c , and suppose for all $r \geq 0$, $H^r(TZ) = 0$. By Lemma 8.2, there exists a complex $X \in D^c(R)$

$$\longrightarrow 0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow 0 \longrightarrow$$

so that Z is a direct summand of πX . We have two maps

$$\pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

so that $bc = 1_Z$. This gives us two maps

$$T\pi X \xrightarrow{Tb} TZ \xrightarrow{Tc} T\pi X$$

with $(Tb)(Tc) = 1$. Given any map

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ),$$

we can form the composite

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ) \xrightarrow{H^0(Tc)} H^0(T\pi X).$$

Of course, $T\pi X$ is just the chain complex

$$\cdots \longrightarrow \sigma^{-1}X^{-1} \longrightarrow \sigma^{-1}X^0 \longrightarrow 0 \longrightarrow$$

and any map from a projective $\sigma^{-1}P$ to $H^0(T\pi X)$ lifts to a map

$$\sigma^{-1}P \longrightarrow T\pi X.$$

By Lemma 8.1, the above map is $T\gamma$, for a (unique) map

$$\pi P \xrightarrow{\gamma} \pi X.$$

Now let μ be the composite

$$\pi P \xrightarrow{\gamma} \pi X \xrightarrow{b} Z.$$

Applying the functor $H^0 \circ T$, we compute $H^0(T\mu)$ to be the composite

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ) \xrightarrow{H^0(Tc)} H^0(T\pi X) \xrightarrow{H^0(Tb)} H^0(TZ),$$

which is nothing other than the map a . \square

Lemma 8.4. *For any f.g. projective $\sigma^{-1}R$ -module M , there is a canonically unique object $\widetilde{M} \in \mathcal{C}^c$ so that*

$$\mathbf{8.4.1.} \quad H^0(T\widetilde{M}) = 0 \text{ for } n \neq 0.$$

$$\mathbf{8.4.2.} \quad H^0(T\widetilde{M}) = M.$$

The functor $H^0(T-)$ is an equivalence of categories between the full subcategory of objects $\widetilde{M} \in \mathcal{C}^c$ and f.g. projective $\sigma^{-1}R$ -modules.

Proof. Let us first prove existence. Let M be a f.g. projective $\sigma^{-1}R$ -module. There exists a $\sigma^{-1}R$ -module N , so that $M \oplus N \cong \{\sigma^{-1}R\}^r$. There is an idempotent $\{\sigma^{-1}R\}^r \longrightarrow \{\sigma^{-1}R\}^r$ which is the map

$$M \oplus N \xrightarrow{1_M \oplus 0_N} M \oplus N.$$

Write this map as $1_M \oplus 0_N : T\pi R^r \longrightarrow T\pi R^r$. By Proposition 5.2, there is a unique lifting $e : \pi R^r \longrightarrow \pi R^r$. The uniqueness of the lifting allows us to easily show that $e^2 = e$. But idempotents split in \mathcal{C} , by Proposition 1.6.8 of [20]. Define \widetilde{M} by splitting the idempotent e .

Then $H^n(T\widetilde{M})$ is computed by splitting the idempotent $H^n(Te)$ on $H^n(\sigma^{-1}R^r)$; this gives us zero when $n \neq 0$, and M when $n = 0$. We have proved the existence of a \widetilde{M} satisfying 8.4.1 and 8.4.2.

Now suppose X is an object of \mathcal{C}^c , and that

- (i) $H^n(TX) = 0$ for $n \neq 0$,
- (ii) $H^0(TX) = M$.

We wish to produce an isomorphism $\widetilde{M} \longrightarrow X$. In any case, we have a map

$$\sigma^{-1}R^r \longrightarrow M = H^0(TX),$$

namely the projection to the direct summand. By Lemma 8.3, there is a map

$$\pi R^r \longrightarrow X,$$

which induces the projection. We may form the composite

$$\widetilde{M} \longrightarrow \pi R^r \longrightarrow X,$$

and it is very easy to check that the map

$$T\widetilde{M} \longrightarrow TX$$

is a homology isomorphism, hence an isomorphism in $D^c(\sigma^{-1}R)$. If we complete $\widetilde{M} \longrightarrow X$ to a triangle in \mathcal{C}^c

$$\widetilde{M} \longrightarrow X \longrightarrow Y \longrightarrow \Sigma\widetilde{M},$$

then $TY = 0$. But by Proposition 5.4 it then follows that $Y = 0$, and $\widetilde{M} \rightarrow X$ is an isomorphism.

Finally it remains to check that $\mathcal{C}(\widetilde{M}, \widetilde{N}) = \text{Hom}_{\sigma^{-1}R}(M, N)$. By the construction of \widetilde{M} and \widetilde{N} as direct summands of πR^r and πR^s , this reduces to knowing that

$$\mathcal{C}(\pi R^r, \pi R^s) = \mathcal{D}(T\pi R^r, T\pi R^s).$$

But we know this from Proposition 5.2. \square

Theorem 8.5. *The map $T : \mathcal{C}^c \rightarrow \mathcal{D}^c$ of Proposition 5.1 induces a K_0 -isomorphism.*

Proof. We have maps of categories

$$\mathcal{P}(\sigma^{-1}R) \xrightarrow{a} \mathcal{C}^c \xrightarrow{T} \mathcal{D}^c$$

with $\mathcal{P}(\sigma^{-1}R)$ the category of f.g. projective $\sigma^{-1}R$ -modules. The map T is given by Proposition 5.1; the map a takes a f.g. projective $\sigma^{-1}R$ -module M to $a(M) = \widetilde{M}$. In K -theory, the composite

$$K_0(\sigma^{-1}R) \rightarrow K_0(\mathcal{C}^c) \rightarrow K_0(\mathcal{D}^c)$$

is clearly an isomorphism. To prove that both maps are isomorphisms, it suffices to show that the map $K_0(a) : K_0(\sigma^{-1}R) \rightarrow K_0(\mathcal{C}^c)$ is onto. This is what we shall do.

Let Z be an object of \mathcal{C}^c . We want to show that its class $[Z] \in K_0(\mathcal{C}^c)$ lies in the image of $K_0(a)$. We shall prove this by induction on the length of TZ . In any case, TZ is an object of $\mathcal{D}^c = \mathcal{D}^c(\sigma^{-1}R)$; it is a bounded complex of f.g. projective $\sigma^{-1}R$ -modules.

Suppose the length of TZ is zero. Replacing Z by a suspension, this means that $H^n(TZ) = 0$ unless $n = 0$. But then $H^0(TZ) = M$ must be a f.g. projective $\sigma^{-1}R$ -module, and by Lemma 8.4 we know that Z is (canonically) isomorphic to \widetilde{M} . Thus Z is in the image of a .

Suppose now that we know the induction hypothesis. We are given $n \geq 0$. We know that if Z is an object of \mathcal{C}^c so that the length of TZ is $\leq n$, then the class $[Z] \in K_0(\mathcal{C}^c)$ lies in the image of $K_0(a)$. Let Z be a complex of length $n+1 \geq 1$. Replacing Z by a suspension, this means that $H^r(TZ) = 0$ unless $-n-1 \leq r \leq 0$. Now $H^0(TZ)$ is a finitely presented $\sigma^{-1}R$ -module; we may choose a f.g. free R -module F , and a surjection $\sigma^{-1}F \rightarrow H^0(TZ)$. By Lemma 8.3, there is a map

$$\pi F \rightarrow Z$$

lifting this surjection. Form the triangle in \mathcal{C}^c

$$\pi F \rightarrow Z \rightarrow Y \rightarrow \Sigma \pi F.$$

It is easily computed that the length of TY is $\leq n$, so by induction $[Y]$ lies in the image of $K_0(a) : K_0(\sigma^{-1}R) \rightarrow K_0(\mathcal{C}^c)$. Clearly $[\pi F] = [\widetilde{\sigma^{-1}F}]$ also lies in the image of $K_0(a)$, and the triangle tells us that $[Z] = [Y] + [\pi F]$. \square

9. T INDUCES A K_1 -ISOMORPHISM

In Proposition 5.1 we produced a triangulated functor of triangulated categories

$$T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R) .$$

In Section 8 we proved that the induced map $K_0(T) : K_0(\mathcal{C}^c) \longrightarrow K_0(\mathcal{D}^c)$ is an isomorphism. The main result of this section is that so is $K_1(T) : K_1(\mathcal{C}^c) \longrightarrow K_1(\mathcal{D}^c)$. First we must address a point concerning Waldhausen K -theory.

Let \mathbf{B} be the category whose objects are all bounded chain complexes of f.g. projective R -modules. The morphisms in \mathbf{B} are the chain maps. The cofibrations are the maps which are split monomorphisms in each degree. The weak equivalences are the homology isomorphisms. Clearly, \mathbf{B} is a model for the triangulated category $\mathcal{B}^c = D^c(R)$.

Let \mathbf{A} be the full subcategory of all objects in \mathbf{B} whose image in \mathcal{B}^c is contained in \mathcal{A} . Then \mathbf{A} is a model for the triangulated category \mathcal{A}^c , and the inclusion $\mathbf{A} \longrightarrow \mathbf{B}$ is a model for the map $\mathcal{A}^c \longrightarrow \mathcal{B}^c$. Let \mathbf{C} be the same category as \mathbf{B} , with the same cofibrations, but different weak equivalences. The weak equivalences in \mathbf{C} are the maps in $\mathbf{C} = \mathbf{B}$ whose mapping cone lies in $\mathbf{A} \subset \mathbf{B}$. We have exact functors of Waldhausen categories

$$\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$$

which induce the maps of triangulated categories

$$\mathcal{A}^c \longrightarrow \mathcal{B}^c \longrightarrow \mathcal{B}^c/\mathcal{A}^c .$$

Let \mathbf{D} be the Waldhausen category of all bounded chain complexes of f.g. projective $\sigma^{-1}R$ -modules. The cofibrations are the maps which are split monomorphisms in each degree. The weak equivalences are the homology isomorphisms. The functor $X \mapsto \sigma^{-1}X$ is clearly an exact functor of Waldhausen categories $\mathbf{B} \longrightarrow \mathbf{D}$. In Proposition 5.2 we showed that it factors as

$$\mathbf{B} \longrightarrow \mathbf{C} \longrightarrow \mathbf{D} .$$

Waldhausen's localization theorem tells us that there is a homotopy fibration in Waldhausen K -theory

$$K(\mathbf{A}) \longrightarrow K(\mathbf{B}) \longrightarrow K(\mathbf{C}) ;$$

somewhat loosely, we have been referring to this homotopy fibration as

$$K(\mathcal{A}^c) \longrightarrow K(\mathcal{B}^c) \longrightarrow K(\mathcal{B}^c/\mathcal{A}^c) .$$

The map we have been calling $K(\mathcal{B}^c/\mathcal{A}^c) \longrightarrow K(\mathcal{C}^c)$ is an isomorphism except possibly on K_0 . Since this section deals only with K_1 , there is no point in explaining what this map is, on the level of Waldhausen models. There is also a map $K(\mathbf{C}) \longrightarrow K(\mathbf{D})$. This is the map we have loosely been referring to as $K(\mathcal{B}^c/\mathcal{A}^c) \longrightarrow K(\mathcal{D}^c)$.

In the proof of Theorem 8.5, we also introduced a functor $\mathcal{P}(\sigma^{-1}R) \longrightarrow \mathcal{C}^c$. We do not know a Waldhausen model for this map. But in the following discussion we shall show that there is an induced map on K_1 . The group $K_1(\sigma^{-1}R)$ is generated by

determinants of automorphisms of free (or projective) modules. Certainly, the following generate: given any projective R -module P , and an automorphism $\phi : \sigma^{-1}P \longrightarrow \sigma^{-1}P$, the determinant of ϕ is an element of $K_1(\sigma^{-1}R)$, and the collection of all determinants of all ϕ 's generates $K_1(\sigma^{-1}R)$. We want to produce a map $K_1(\sigma^{-1}R) \longrightarrow K_1(\mathbf{C})$; to define the map, it suffices to say what it does on all ϕ 's as above.

To define what the map does to ϕ , let us remind ourselves that the zero-space of the spectrum $K(\mathbf{C})$ has a Gillet-Grayson model (see [17]) , which we denote $GG(\mathbf{C})$. That is, there is a homotopy equivalence

$$GG(\mathbf{C}) \simeq \Omega^\infty K(\mathbf{C}) .$$

The space $GG(\mathbf{C})$ is an H -space, and hence

$$K_1(\mathbf{C}) = \pi_1 K(\mathbf{C}) = \pi_1 GG(\mathbf{C}) = H_1 GG(\mathbf{C}) .$$

Starting with an automorphism $\phi : \sigma^{-1}P \longrightarrow \sigma^{-1}P$, we need to produce a class in the first homology group $H_1 GG(\mathbf{C})$.

We note that, by Proposition 5.2, $\phi : \sigma^{-1}P \longrightarrow \sigma^{-1}P$ corresponds to a unique automorphism

$$\varphi : \pi P \longrightarrow \pi P .$$

This is an automorphism defined in $\mathcal{B}^c/\mathcal{A}^c$, and \mathbf{C} is a Waldhausen model for $\mathcal{B}^c/\mathcal{A}^c$. It follows that there exist weak equivalences $a : Q \longrightarrow P$ and $b : Q \longrightarrow P$, with $\varphi = ab^{-1}$. But then P and Q are 0-cells in the Gillet-Grayson model $GG(\mathbf{C})$. The weak equivalences $a : Q \longrightarrow P$ and $b : Q \longrightarrow P$ are 1-cells. Now $[a] - [b]$ is a cycle, that is an element in $H_1(GG(\mathbf{C})) = \pi_1(GG(\mathbf{C})) = K_1(\mathbf{C})$. We leave it to the reader to check that the map sending ϕ to $[a] - [b]$ extends to a well-defined homomorphism $K_1(\sigma^{-1}R) \longrightarrow K_1(\mathbf{C})$.

The composite $K_1(\sigma^{-1}R) \longrightarrow K_1(\mathbf{C}) \longrightarrow K_1(\mathbf{D})$ is easily seen to be an isomorphism; in our looser notation, it is the map

$$K_1(\sigma^{-1}R) \longrightarrow K_1(\mathcal{B}^c/\mathcal{A}^c) = K_1(\mathcal{C}^c) \longrightarrow K_1(\mathcal{D}^c) .$$

To prove that both maps are isomorphisms it suffices therefore to check that $K_1(\sigma^{-1}R) \longrightarrow K_1(\mathbf{C})$ is epi. We have a localization exact sequence

$$K_1(\mathbf{B}) \longrightarrow K_1(\mathbf{C}) \longrightarrow K_0(\mathbf{A}) \longrightarrow K_0(\mathbf{B}) .$$

Note that $K(\mathbf{B}) = K(D^c(R)) = K(R)$. The composite $K_1(R) \longrightarrow K_1(\sigma^{-1}R) \longrightarrow K_1(\mathbf{C})$ is easily computed to agree with the natural $K_1(R) = K_1(\mathbf{B}) \longrightarrow K_1(\mathbf{C})$; we deduce a commutative diagram where the bottom row is exact

$$\begin{array}{ccccccc} K_1(R) & \longrightarrow & K_1(\sigma^{-1}R) & & & & \\ 1 \downarrow & & \psi \downarrow & & & & \\ K_1(R) & \longrightarrow & K_1(\mathbf{C}) & \longrightarrow & K_0(\mathbf{A}) & \longrightarrow & K_0(R) . \end{array}$$

To prove ψ epi, it suffices to show that the composite

$$\begin{array}{ccc} K_1(\sigma^{-1}R) & & \\ \psi \downarrow & & \\ K_1(\mathbf{C}) & \longrightarrow & K_0(\mathbf{A}) \end{array}$$

surjects to the kernel of $K_0(\mathbf{A}) \longrightarrow K_0(R)$. But the composite is easy to compute. Take an automorphism $\phi : \sigma^{-1}R \otimes P \longrightarrow \sigma^{-1}R \otimes P$ as above, which corresponds as above to an automorphism

$$\varphi : \pi P \longrightarrow \pi P .$$

Choose weak equivalences $a : Q \longrightarrow P$ and $b : Q \longrightarrow P$, with $\varphi = ab^{-1}$. Then ϕ gets sent to $[A] - [B]$, where

$$\begin{aligned} A : \dots \longrightarrow 0 \longrightarrow Q \xrightarrow{a} P \longrightarrow 0 \longrightarrow \dots , \\ B : \dots \longrightarrow 0 \longrightarrow Q \xrightarrow{b} P \longrightarrow 0 \longrightarrow \dots . \end{aligned}$$

It will therefore suffice to show that every element in the kernel of the map $K_0(\mathbf{A}) \longrightarrow K_0(R)$ can be expressed as a difference $[A] - [B]$, as above. We shall prove something stronger.

Theorem 9.1. *Every element in $K_0(\mathbf{A}) = K_0(\mathcal{A}^c)$ is a linear combination of complexes of length ≤ 1 . That is, it may be written as $\sum \pm[A_i]$, with $A_i \in \mathcal{A}^c$ being complexes of f.g. projective R -modules of the form*

$$\dots \longrightarrow 0 \longrightarrow X \longrightarrow Y \longrightarrow 0 \longrightarrow \dots .$$

□

Note that with K_0 , it makes no difference whether we compute K_0 of a triangulated category, or K_0 of its model; from now on we can forget all about models. Before the proof of Theorem 9.1, let us state the main corollary

Corollary 9.2. *Every object in the kernel of the map $K_0(\mathcal{A}^c) \longrightarrow K_0(\mathcal{B}^c) = K_0(R)$ is of the form $[A] - [B]$, where A is a complex*

$$\dots \longrightarrow 0 \longrightarrow Q \xrightarrow{a} P \longrightarrow 0 \longrightarrow \dots$$

and B is a complex

$$\dots \longrightarrow 0 \longrightarrow Q \xrightarrow{b} P \longrightarrow 0 \longrightarrow \dots$$

By the discussion preceding Theorem 9.1, this means that the map $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ induces a K_1 -isomorphism.

Proof that Corollary 9.2 follows from Theorem 9.1. Suppose we have an element of the kernel of the map $K_0(\mathcal{A}^c) \longrightarrow K_0(\mathcal{B}^c) = K_0(R)$. By Theorem 9.1, just by virtue of being an element of $K_0(\mathcal{A}^c)$, it has an expression as $\sum \pm[A_i]$, with $A_i \in \mathcal{A}^c$ being complexes of f.g. projective R -modules of the form

$$\dots \longrightarrow 0 \longrightarrow X_i \longrightarrow Y_i \longrightarrow 0 \longrightarrow \dots$$

Recalling that $[\Sigma A_i] = -[A_i]$, up to changing signs in the sum we may assume that all the X_i are in degree -1 , all the Y_i in degree 0 . Collecting together all the terms of equal sign, we may rewrite the sum as

$$[\oplus A_i] - [\oplus B_j].$$

That is, we have an element $[A] - [B]$ in the kernel of $K_0(\mathcal{A}^c) \longrightarrow K_0(\mathcal{B}^c) = K_0(R)$, where A, B are complexes of the form

$$A : \dots \longrightarrow 0 \longrightarrow A^{-1} \xrightarrow{a} A^0 \longrightarrow 0 \longrightarrow \dots,$$

$$B : \dots \longrightarrow 0 \longrightarrow B^{-1} \xrightarrow{b} B^0 \longrightarrow 0 \longrightarrow \dots.$$

The fact that $[A] - [B]$ lies in the kernel of the map $K_0(\mathcal{A}^c) \longrightarrow K_0(\mathcal{B}^c) = K_0(R)$ tells us that, in $K_0(R)$, there is an identity

$$[A^{-1}] + [B^0] = [B^{-1}] + [A^0].$$

This in turn says that there is a projective R -module X , and an isomorphism

$$A^{-1} \oplus B^0 \oplus X \cong B^{-1} \oplus A^0 \oplus X.$$

The object $[A] \in \mathcal{A}^c$ is isomorphic to the complex

$$\dots \longrightarrow 0 \longrightarrow A^{-1} \oplus B^0 \oplus X \xrightarrow{a \oplus 1_{B^0} \oplus 1_X} A^0 \oplus B^0 \oplus X \longrightarrow 0 \longrightarrow \dots$$

while the object $[B] \in \mathcal{A}^c$ is isomorphic to the complex

$$\dots \longrightarrow 0 \longrightarrow B^{-1} \oplus A^0 \oplus X \xrightarrow{b \oplus 1_{A^0} \oplus 1_X} B^0 \oplus A^0 \oplus X \longrightarrow 0 \longrightarrow \dots$$

Put $Q = A^{-1} \oplus B^0 \oplus X \cong B^{-1} \oplus A^0 \oplus X$, and $P = A^0 \oplus B^0 \oplus X$. Then A is isomorphic in \mathcal{A}^c to a complex

$$\dots \longrightarrow 0 \longrightarrow Q \xrightarrow{\alpha} P \longrightarrow 0 \longrightarrow \dots$$

and B is isomorphic in \mathcal{A}^c to a complex

$$\dots \longrightarrow 0 \longrightarrow Q \xrightarrow{\beta} P \longrightarrow 0 \longrightarrow \dots$$

as required. □

Proof of Theorem 9.1. It remains to prove Theorem 9.1. Let X be an object of \mathcal{A}^c . We need to show that the class $[X]$ in $K_0(\mathcal{A}^c)$ can be written as a linear combination of classes of objects of length ≤ 1 .

Because $X \in \mathcal{A}^c \subset \mathcal{B}^c$, we have that X is a bounded complex of f.g. projective R -modules. Suspending suitably, we may assume it has the form

$$\longrightarrow 0 \longrightarrow X^{-m} \longrightarrow X^{-m+1} \longrightarrow \dots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow 0 \longrightarrow$$

If $m \leq 1$ we are done; the complex has length ≤ 1 . The proof is by induction. Assume we are given an integer $n \geq 1$. Assume further that, for every $X \in \mathcal{A}^c$ of length $m \leq n$, $[X]$ is equal in $K_0(\mathcal{A}^c)$ to a linear combination of complexes of length ≤ 1 . Take a complex X as above, with $m = n + 1 \geq 2$. We need to show that it can also be expressed as a linear combination of complexes of length ≤ 1 .

By Lemma 4.8, we have that every object of \mathcal{A}^c is isomorphic to a direct summand of an object in \mathcal{S} , with \mathcal{S} defined as in Definition 4.4. Choose a chain complex $Y \in \mathcal{S}$ and maps $X \longrightarrow Y \longrightarrow X$ composing to the identity on X . Clearly, the map $H^0(Y) \longrightarrow H^0(X)$ must be surjective.

By Lemma 4.6, there exist exists a triangle

$$U \longrightarrow Y \longrightarrow V \longrightarrow \Sigma U$$

with $U \in \mathcal{S}[1, \infty)$ and $V \in \mathcal{S}(-\infty, 1]$. The composite

$$U \longrightarrow Y \longrightarrow X$$

is a map from $U \in \mathcal{S}[1, \infty)$ to $X \in \mathcal{B}^{\leq 0}$, which must vanish. It follow that $Y \longrightarrow X$ factors as $Y \longrightarrow V \longrightarrow X$. And since $H^0(Y) \longrightarrow H^0(X)$ is epi and factors through $H^0(V)$, we deduce that $H^0(V) \longrightarrow H^0(X)$ must be epi. Replacing Y by V , we may assume $Y \in \mathcal{S}(-\infty, 1]$.

Next we apply Lemma 4.6 again, this time to deduce that $Y \in \mathcal{S}(-\infty, 1]$ can be expressed as the mapping cone on a map $U \longrightarrow V$, with $U \in \mathcal{S}(-\infty, 0]$ and $V \in \mathcal{S}[-1, 1]$. There is a triangle

$$U \longrightarrow V \longrightarrow Y \longrightarrow \Sigma U ,$$

hence an exact sequence

$$H^0(V) \longrightarrow H^0(Y) \longrightarrow H^1(U) = 0 .$$

The map $H^0(V) \longrightarrow H^0(Y) \longrightarrow H^0(X)$ is the composite of two epis, hence is epi. Replacing $Y \longrightarrow X$ by the composite $V \longrightarrow Y \longrightarrow X$, we may assume $Y \in \mathcal{S}[-1, 1]$.

The last time we apply Lemma 4.6 is to express Y as the mapping cone of a map $U \longrightarrow Z$, with $U \in \mathcal{S}[0, 1]$ and $Z \in \mathcal{S}[0, 1]$. The only observations we wish to make is that $H^1(Z) \longrightarrow H^1(Y)$ is epi, and that in $K_0(\mathcal{A}^c)$, $[Z]$ and $[Y] = [Z] - [U]$ are both linear combinations of objects in \mathcal{A} of length ≤ 1 . Let us summarize: we have constructed maps $Z \longrightarrow Y \longrightarrow X$, with

- (i) $Z \in \mathcal{S}[0, 1]$, $Y \in \mathcal{S}[-1, 1]$,
- (ii) $H^0(Y) \longrightarrow H^0(X)$ epi,
- (iii) $H^1(Z) \longrightarrow H^1(Y)$ epi,
- (iv) both $[Y]$ and $[Z]$ are linear combinations of objects in \mathcal{A} of length ≤ 1 .

Form the mapping cone on the map $Y \longrightarrow X$, to obtain a triangle

$$Y \longrightarrow X \longrightarrow X' \longrightarrow Y .$$

Since $Y \in \mathcal{S}[-1, 1]$ while $X \in \mathcal{A}^c$ is supported on the interval $[-m, 0]$ with $m \geq 2$, the mapping cone X' is an object of \mathcal{A}^c supported in $[-m, 0]$. The long exact sequence in homology gives

$$H^0(Y) \longrightarrow H^0(X) \longrightarrow H^0(X') \longrightarrow H^1(Y) \longrightarrow H^1(X) .$$

We have $H^1(X) = 0$, while $H^0(Y) \rightarrow H^0(X)$ is an epimorphism. Hence $H^0(X') = H^1(Y)$. But we know that the map $H^1(Z) \rightarrow H^1(Y)$ is an epimorphism, by (iii). And Z is a complex of the form

$$\dots \rightarrow 0 \rightarrow Z^0 \rightarrow Z^1 \rightarrow 0 \rightarrow \dots$$

that is a complex of length ≤ 1 . It follows that we can extend the epimorphism $\beta : H^1(Z) \rightarrow H^0(X')$ to a map from the presentation Z ; there is a map $\Sigma Z \rightarrow X'$, inducing β in H^0 . We may form the mapping cone, obtaining a triangle

$$\Sigma Z \longrightarrow X' \longrightarrow X'' \longrightarrow \Sigma^2 Z .$$

Since $\Sigma Z \in \mathcal{S}[-1, 0]$ and X' is supported on $[-m, 0]$, we conclude that X'' is supported on $[-m, 0]$. But now the long exact homology sequence

$$H^0(\Sigma Z) \xrightarrow{\alpha} H^0(X') \longrightarrow H^0(X'') \longrightarrow H^1(\Sigma Z)$$

has α surjective, while $H^1(\Sigma Z) = 0$. We conclude that $H^0(X'') = 0$. The complex X'' is supported in the interval $[m, -1]$. By induction, its class in $K_0(\mathcal{A}^c)$ is a linear combination of complexes of length ≤ 1 .

But now the triangles above give the identities

$$[X] = [X'] + [Y], \quad [X'] = [X''] - [Z]$$

and hence $[X] = [X''] + [Y] - [Z]$, and all the terms on the right may be expressed as linear combinations of complexes of length ≤ 1 . \square

10. T IS AN EQUIVALENCE IF AND ONLY IF THE TOR-GROUPS VANISH

In Proposition 5.1 we constructed a functor

$$T : \mathcal{C}^c = (D(R)/D(R, \sigma))^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R) .$$

In Sections 8 and 9 we showed that T induces an isomorphism in K_0 and K_1 . For higher K -theory, the useful result we have is a necessary and sufficient conditions for the functor T to be an equivalence of categories. An equivalence of categories trivially induces an isomorphism in K -theory. It is very easy to see a necessary condition:

Lemma 10.1. *If $T : \mathcal{C}^c \rightarrow \mathcal{D}^c$ is an equivalence then, for all $n \neq 0$, $H^n(G\pi R) = 0$.*

Proof. Suppose T is an equivalence. We have isomorphisms

$$\begin{aligned} H^n(G\pi R) &= \mathcal{B}^c(R, \Sigma^n G\pi R) \\ &= \mathcal{C}^c(\pi R, \Sigma^n \pi R) \\ &= \mathcal{D}(T\pi R, \Sigma^n T\pi R) . \end{aligned}$$

But $T\pi R = \sigma^{-1}R$, and in the category $\mathcal{D} = D(\sigma^{-1}R)$, the maps $\sigma^{-1}R \rightarrow \Sigma^n \sigma^{-1}R$ all vanish, whenever $n \neq 0$. \square

Remark 10.2. The main result of this section is that the converse of Lemma 10.1 holds. The condition is also sufficient. \square

Proposition 10.3. *The groups $\{H^n(G\pi R), n \neq 0\}$ all vanish if and only if the groups $\{\mathrm{Tor}_n^R(\sigma^{-1}R, \sigma^{-1}R), n \neq 0\}$ all vanish.*

Proof. By Lemma 3.17, the groups $H^n(G\pi R)$ vanish when $n > 0$, while the groups $\mathrm{Tor}_n^R(\sigma^{-1}R, \sigma^{-1}R)$ clearly vanish when $n < 0$. By Corollary 3.27, $\mathrm{Tor}_1^R(\sigma^{-1}R, \sigma^{-1}R) = 0$. By Corollary 3.31, the smallest non-zero integer n for which $H^{-n}(G\pi R) \neq 0$ is equals the smallest integer $n \neq -1$ for which $\mathrm{Tor}_{n+1}^R(\sigma^{-1}R, \sigma^{-1}R) \neq 0$. \square

Corollary 10.4. *Suppose $H^n(G\pi R) = 0$ for all $n \neq 0$. Put $S = \sigma^{-1}R$. We assert*

10.4.1. *$S \otimes_R S = S$ via the multiplication map, and*

10.4.2. *$\mathrm{Tor}_i^R(S, S) = 0$ for all $i > 0$.*

Proof. 10.4.2 is immediate from Proposition 10.3, while 10.4.1 follows from Corollary 3.27. \square

Next we shall formally study the consequences of 10.4.1 and 10.4.2. Let us begin with an easy general observation, about natural transformations of functors respecting coproducts.

Lemma 10.5. *Let \mathcal{S} and \mathcal{T} be triangulated categories, let F and G be triangulated functors $\mathcal{S} \rightarrow \mathcal{T}$, and let $\rho : F \Rightarrow G$ be a natural transformation commuting with the suspension. Define the full subcategory $\mathcal{J} \subset \mathcal{S}$ by the formula*

$$\mathrm{Ob}(\mathcal{J}) = \{x \in \mathrm{Ob}(\mathcal{S}) \mid \rho_x \text{ is an isomorphism}\}.$$

Then the category \mathcal{J} is triangulated. If furthermore both F and G commute with coproducts, then \mathcal{J} contains all coproducts in \mathcal{S} of its objects.

Proof. Left to the reader. \square

Lemma 10.6. *Let $R \rightarrow S$ be a ring homomorphism such that R and S satisfy the hypotheses 10.4.1 and 10.4.2; we remind the reader*

10.4.1: *$S \otimes_R S = S$ via the multiplication map, and*

10.4.2: *$\mathrm{Tor}_i^R(S, S) = 0$ for all $i > 0$.*

Define the functor

$$\Theta : D(S) \rightarrow D(S) ; X \mapsto S^L \otimes_R X.$$

Multiplication defines a natural transformation $\mu : \Theta \Rightarrow 1$. We assert that μ is an isomorphism.

Proof. The functors Θ and 1 are triangulated functors respecting coproducts, and the natural transformation μ commutes with the suspension. By Lemma 10.5, the full subcategory of all $x \in D(S)$ for which μ is an isomorphism is triangulated category closed under coproducts. By 10.4.1 and 10.4.2, it also contains the object $S \in D(S)$. From Lemma 3.5, it follows that μ_x is an isomorphism for all $x \in D(S)$. \square

Theorem 10.7. *Let $\alpha : R \longrightarrow S$ be a ring homomorphism such that R and S satisfy the hypotheses 10.4.1 and 10.4.2; we remind the reader*

10.4.1: $S \otimes_R S = S$ via the multiplication map, and

10.4.2: $\mathrm{Tor}_i^R(S, S) = 0$ for all $i > 0$.

Put $\mathcal{B} = D(R)$, $\mathcal{D} = D(S)$. There is a functor $\pi : \mathcal{B} \longrightarrow \mathcal{D}$ taking $X \in D(R)$ to $S^L \otimes_R X \in D(S)$. The functor π has a fully faithful right adjoint $G : \mathcal{D} \longrightarrow \mathcal{B}$. The unit of adjunction $\eta : 1 \Longrightarrow G\pi$ is identified by saying that the two maps below are naturally isomorphic

$$X \xrightarrow{\eta_X} G\pi X \quad , \quad R^L \otimes_R X \xrightarrow{\alpha \otimes X} S^L \otimes_R X .$$

The functor G respects coproducts. Finally, there exists a thick subcategory $\mathcal{A} \subset \mathcal{B}$ so that the pair of adjoints π, G are a Bousfield localization for the pair $\mathcal{A} \subset \mathcal{B}$.

Proof. The functor

$$\pi : \mathcal{B} = D(R) \longrightarrow \mathcal{D} = D(S) ; X \mapsto S^L \otimes_R X$$

is left adjoint to the forgetful functor $G : \mathcal{D} \longrightarrow \mathcal{B}$ taking a complex of S -modules and just viewing it as a complex of R -modules. This is just the derived category version of the classical fact that for any R -module M and S -module N

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_S(S \otimes_R M, N) .$$

The functor G clearly respects coproducts, and the unit of adjunction clearly identifies as

$$R^L \otimes_R X \xrightarrow{\alpha \otimes X} S^L \otimes_R X .$$

The first statement that requires proof is that G is fully faithful. Suppose therefore that X and Y are complexes of S -modules. Then

$$\begin{aligned} \mathrm{Hom}_{D(R)}(X, Y) &= \mathrm{Hom}_{D(S)}(S^L \otimes_R X, Y) && \text{because } \pi \text{ is left adjoint to } G \\ &= \mathrm{Hom}_{D(S)}(X, Y) && \text{by Lemma 10.6, } S^L \otimes_R X = X . \end{aligned}$$

Since the homomorphisms are the same in $D(S) = \mathcal{D}$ as in $D(R) = \mathcal{B}$, the inclusion G is fully faithful.

This means that the fully faithful inclusion $G : \mathcal{D} \longrightarrow \mathcal{B}$ has a left adjoint, and hence, by Proposition 9.1.18 in [20], a Bousfield colocalization exists for the pair $\mathcal{D} \subset \mathcal{B}$. Put $\mathcal{A} = \mathcal{D}^\perp$, in the notation of Chapter 9 loc. cit. By Corollary 9.1.14 loc. cit. a Bousfield localization functor exists for the pair $\mathcal{A} \subset \mathcal{B}$, with $\mathcal{D} = {}^\perp \mathcal{A}$, in other words $G : \mathcal{D} \longrightarrow \mathcal{B}$ is the right adjoint of the quotient map. The quotient map must be the left adjoint of G , that is $\pi : D(R) \longrightarrow D(S)$, as above. \square

It remains to apply Theorem 10.7 to the special case of Corollary 10.4, that is where $S = \sigma^{-1}R$.

Theorem 10.8. *Let R be a ring, σ a set of morphisms of f.g. projective R -modules. Set*

$$\mathcal{A} = D(R, \sigma), \mathcal{B} = D(R), \mathcal{C} = D(R)/D(R, \sigma), \mathcal{D} = D(\sigma^{-1}R)$$

with $\mathcal{A} \subset \mathcal{B}$ the subcategory generated by σ . Let $\pi : \mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}/\mathcal{A}$ be the projection, $G : \mathcal{C} \longrightarrow \mathcal{B}$ its right adjoint. The following conditions are equivalent :

- (i) $H^n(G\pi R) = 0$ for all $n \neq 0$,
- (ii) $\text{Tor}_i^R(\sigma^{-1}R, \sigma^{-1}R) = 0$ for $i \geq 1$,
- (iii) *the natural functor $T : \mathcal{C} \longrightarrow \mathcal{D}$, of Proposition 5.1, is an equivalence of categories,*
- (iv) *the natural functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$ is an equivalence of categories.*

Proof. The equivalence of (i) and (ii) is given by Proposition 10.3.

The implication (iv) \implies (i) is Lemma 10.1.

The implication (iii) \implies (iv) is obvious.

It remains to prove (ii) \implies (iii). The composite $T\pi$ is the functor $X \mapsto \{\sigma^{-1}R\}^L \otimes_R X$, and by (ii) and Theorem 10.7 we know that it is a projection $\mathcal{B} \longrightarrow \mathcal{B}/{}^\perp \mathcal{D}$. Suppose we could show that ${}^\perp \mathcal{D} = \mathcal{A}$. Then both $T\pi$ and π would be identified as the projection $\mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A} = \mathcal{B}/{}^\perp \mathcal{D}$, and by the universal property of the Verdier quotient, T would be an equivalence. We are reduced to showing ${}^\perp \mathcal{D} = \mathcal{A}$.

The category ${}^\perp \mathcal{D}$ is very explicit: it is the kernel of the functor $T\pi$. That is, ${}^\perp \mathcal{D}$ is the collection of all $X \in \mathcal{B}$ with $\{\sigma^{-1}R\}^L \otimes_R X = 0$. By Lemma 3.20, $\mathcal{A} \subset {}^\perp \mathcal{D}$. We must prove the reverse inclusion. Suppose therefore that $\{\sigma^{-1}R\}^L \otimes_R X = 0$; we need to show that $X \in \mathcal{A}$.

We may form a triangle

$$a \longrightarrow X \xrightarrow{\eta_X} G\pi X \longrightarrow \Sigma a.$$

By 2.6.5 we have that $a \in \mathcal{A}$, and hence $\{\sigma^{-1}R\}^L \otimes_R a = 0$. We are assuming $\{\sigma^{-1}R\}^L \otimes_R X = 0$; the triangle tells us that $\{\sigma^{-1}R\}^L \otimes_R G\pi X = 0$. We want to prove that $X \in \mathcal{A}$, in other words we want to prove that $G\pi X = 0$. It clearly suffices to prove that $G\pi X$ is isomorphic to $\{\sigma^{-1}R\}^L \otimes_R G\pi X$. We shall prove, more precisely, that the natural map

$$R^L \otimes_R G\pi X \xrightarrow{f^L \otimes_R 1_{G\pi}} \{\sigma^{-1}R\}^L \otimes_R G\pi X$$

is an isomorphism, where $f : R \longrightarrow \sigma^{-1}R$ is the natural map $\eta_R : R \longrightarrow H^0(G\pi R) = G\pi R$.

The map $f^L \otimes_R 1_{G\pi}$ is a natural transformation between triangulated functors respecting coproducts, and $f^L \otimes_R 1_{G\pi}$ commutes with the suspension. Form the full subcategory \mathcal{J} given by

$$\text{Ob}(\mathcal{J}) = \{x \in \text{Ob}(\mathcal{D}) \mid f^L \otimes_R 1_{G\pi x} \text{ is an isomorphism}\}.$$

We wish to show that \mathcal{J} is all of $\mathcal{B} = D(R)$. By Lemma 10.5, we know that \mathcal{J} is a triangulated subcategory of \mathcal{B} , closed under coproducts. By Lemma 3.5, it suffices to

show that $R \in \mathcal{J}$. But $f^L \otimes_R 1_{G\pi R}$ is the map $f^L \otimes_R 1 : \sigma^{-1}R \longrightarrow \{\sigma^{-1}R\}^L \otimes_R \{\sigma^{-1}R\}$. Now (ii) tells us that the multiplication map $\mu : \{\sigma^{-1}R\}^L \otimes_R \{\sigma^{-1}R\} \longrightarrow \sigma^{-1}R$ is a homology isomorphism, hence an isomorphism in the derived category. It is clear that the composite

$$\sigma^{-1}R \xrightarrow{f^L \otimes_R 1} \{\sigma^{-1}R\}^L \otimes_R \{\sigma^{-1}R\} \xrightarrow{\mu} \sigma^{-1}R$$

is the identity, forcing $f^L \otimes_R 1$ to be the two-sided inverse of the invertible map μ . \square

As in the Introduction :

Definition 10.9. A noncommutative localization $\sigma^{-1}R$ is *stably flat over R* if it satisfies the equivalent conditions of Theorem 10.8. \square

Theorem 10.10. *Suppose we are given a chain complex $D \in D^c(\sigma^{-1}R)$, that is a bounded chain complex of f.g. projective $\sigma^{-1}R$ modules. Suppose every module in the chain complex is induced; that is, they are all of the form $\sigma^{-1}P$, with P a finitely generated projective R -module. If $\sigma^{-1}R$ is stably flat over R then there exists a complex $C \in D^c(R)$ and a homotopy equivalence $D \cong \sigma^{-1}C$.*

Proof. We are assuming that $\sigma^{-1}R$ is stably flat over R . By Theorem 10.8 this means that the functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ is an equivalence. By 3.9.4, the map $\mathcal{B}^c/\mathcal{A}^c \longrightarrow \mathcal{C}^c = \mathcal{D}^c$ is fully faithful, and $\mathcal{C}^c = \mathcal{D}^c$ is the smallest thick subcategory containing $\mathcal{B}^c/\mathcal{A}^c$. Now note that the objects of the Verdier quotient $\mathcal{B}^c/\mathcal{A}^c$ are the same as the objects of \mathcal{B}^c , and the projection map $\pi : \mathcal{B}^c \longrightarrow \mathcal{B}^c/\mathcal{A}^c$ induces a surjective map in K_0 . Combining this with Proposition 4.5.11 of [20], an object $D \in \mathcal{D}^c = D^c(\sigma^{-1}R)$ is in the image of the functor $T\pi : \mathcal{B}^c \longrightarrow \mathcal{D}^c$ if and only if the class of $[D]$ in $K_0(\mathcal{D}^c)$ lies in the image of the map $K_0(T\pi) : K_0(\mathcal{B}^c) \longrightarrow K_0(\mathcal{D}^c)$. The functor $T\pi$ is by definition the functor taking a complex $C \in \mathcal{B}^c = D^c(R)$ to $\sigma^{-1}C \in \mathcal{D}^c = D^c(\sigma^{-1}R)$. We chose our complex D to be a complex of induced modules. The class of $[D]$ most certainly lies in the image of $K_0(T\pi) : K_0(\mathcal{B}^c) \longrightarrow K_0(\mathcal{D}^c)$. We conclude that there exists a C with D isomorphic in $D^c(\sigma^{-1}R)$ to $\sigma^{-1}C$. \square

Theorem 10.11. *If $\sigma^{-1}R$ is stably flat over R then the functor $T : \mathcal{C}^c \longrightarrow \mathcal{D}^c = D^c(\sigma^{-1}R)$ induces isomorphisms*

$$T : K_*(\mathcal{C}^c) = K_*(D^c(R)/D^c(R, \sigma)) \longrightarrow K_*(\mathcal{D}^c) = K_*(\sigma^{-1}R)$$

and there is a localization exact sequence in algebraic K -theory

$$\dots \longrightarrow K_n(R) \longrightarrow K_n(\sigma^{-1}R) \longrightarrow K_n(R, \sigma) \longrightarrow K_{n-1}(R) \longrightarrow \dots$$

Proof. Combine Theorems 7.4 and 10.8. \square

In Theorem 10.10 we saw that, if $\sigma^{-1}R$ is stably flat over R and $D \in D^c(\sigma^{-1}R)$ is a complex of induced modules, then there exists a complex $C \in D^c(R)$ and a homotopy equivalence $\sigma^{-1}C \cong D$. Now we want a more precise version of this.

Proposition 10.12. *If $C \in D^c(R)$ is a bounded complex of f.g. projective R -modules, and if $\sigma^{-1}C$ is homotopy equivalent to a complex $D \in D^c(\sigma^{-1}R)$ vanishing outside an interval $[0, n]$ with $n \geq 1$, then there exists a $B \in D^c(R)$, vanishing outside $[0, n]$, with $\sigma^{-1}B$ homotopy equivalent to $D \cong \sigma^{-1}C$.*

Proof. We need to show that C can be shortened. Suppose therefore that C is the complex

$$\cdots \longrightarrow 0 \longrightarrow C^{-1} \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0 \longrightarrow \cdots$$

and assume that there is a homotopy equivalence of $\sigma^{-1}C$ with a shorter complex, that is a commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & \sigma^{-1}C^{-1} & \xrightarrow{\partial} & \sigma^{-1}C^0 & \longrightarrow & \cdots & \longrightarrow & \sigma^{-1}C^n & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & \sigma^{-1}C^{-1} & \xrightarrow{\partial} & \sigma^{-1}C^0 & \longrightarrow & \cdots & \longrightarrow & \sigma^{-1}C^n & \longrightarrow & 0 & \longrightarrow \end{array}$$

so that the composite is homotopic to the identity. In particular, there is a map $d : \sigma^{-1}C^0 \longrightarrow \sigma^{-1}C^{-1}$ so that $d\partial : \sigma^{-1}C^{-1} \longrightarrow \sigma^{-1}C^{-1}$ is the identity.

By Proposition 5.2, the map $d : \sigma^{-1}C^0 \longrightarrow \sigma^{-1}C^{-1}$ lifts uniquely to a map $d' : \pi C^0 \longrightarrow \pi C^{-1}$. By Proposition 4.11, the map d' can be represented as $\alpha^{-1}\beta$, where α and β are, respectively, the chain maps

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^{-1} & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & g \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

The fact that $\sigma^{-1}\alpha$ is an equivalence tells us that the map $\sigma^{-1}r : \sigma^{-1}X \longrightarrow \sigma^{-1}Y$ is injective, with cokernel $\sigma^{-1}C^{-1}$. The fact that $\alpha^{-1}\beta$ agrees with d' means that the composite

$$\sigma^{-1}C^0 \xrightarrow{\sigma^{-1}g} \sigma^{-1}Y \longrightarrow \text{Coker}(\sigma^{-1}r)$$

is just the map $d : \sigma^{-1}C^0 \longrightarrow \sigma^{-1}C^{-1}$. Let B be the chain complex

$$\longrightarrow 0 \longrightarrow C^0 \oplus X \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} C^1 \oplus Y \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0 \longrightarrow$$

There is a natural map $f : B \rightarrow C$, and $\sigma^{-1}f$ is a homology isomorphism of bounded complexes of projectives, hence a homotopy equivalence. Thus $\sigma^{-1}B$ is homotopy equivalent to $\sigma^{-1}C \cong D$.

This permits us to shorten on the left. Shortening the complex on the right is dual. \square

11. TORSION MODULES

Until now all our theorems were general, in the sense that we did not impose any restrictions on the ring R or on the set of maps σ .

Hypothesis 11.1. In this section, we assume that all the morphisms in σ are injections. \square

The main theorem of this section is that, under the above restriction, the higher Waldhausen K -theory of the triangulated category $\mathcal{A}^c = D^c(R, \sigma)$ agrees with the higher Quillen K -theory of the exact category $\mathcal{E} = H(R, \sigma)$ of σ -torsion R -modules with projective dimension ≤ 1 .

Proposition 11.2. *If $R \rightarrow \sigma^{-1}R$ is an injection then every $s : P \rightarrow Q$ in σ is an injection, i.e. Hypothesis 11.1 is satisfied.*

Proof. Since $R \rightarrow \sigma^{-1}R$ is a monomorphism and P is projective and therefore flat, we deduce that

$$R \otimes_R P \longrightarrow \{\sigma^{-1}R\} \otimes_R P$$

is a monomorphism. In other words, the map $P \rightarrow \sigma^{-1}P$ is mono. Consider the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{s} & Q \\ \downarrow & & \downarrow \\ \sigma^{-1}P & \xrightarrow{\sigma^{-1}s} & \sigma^{-1}Q \end{array}$$

By the above, $P \rightarrow \sigma^{-1}P$ is an injection. Now $\sigma^{-1}s : \sigma^{-1}P \rightarrow \sigma^{-1}Q$ is an isomorphism. From the commutativity of the square we deduce that $s : P \rightarrow Q$ is an injection. \square

Example 11.3. The converse of Proposition 11.2 does not hold in general. The set $\sigma = \{0 \rightarrow R\}$ satisfies Hypothesis 11.1, but $R \rightarrow \sigma^{-1}R = 0$ is not injective. \square

Definition 11.4. An (R, σ) -module T is an R -module which admits a f.g. projective R -module resolution of length 1

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

with $\sigma^{-1}s : \sigma^{-1}P \longrightarrow \sigma^{-1}Q$ a $\sigma^{-1}R$ -module isomorphism. Let $\mathcal{E} = H(R, \sigma)$ be the full subcategory of the category of R -modules with objects the (R, σ) -modules.

□

Remark 11.5. An R -module T is an (R, σ) -module if and only if :

- (i) $\text{Tor}_i^R(\sigma^{-1}R, T) = 0$ for all $i \in \mathbb{Z}$. In particular, $\sigma^{-1}T = \text{Tor}_0^R(\sigma^{-1}R, T) = 0$.
- (ii) T has projective dimension ≤ 1 .
- (iii) T is finitely presented.

□

Lemma 11.6. *The category \mathcal{E} of (R, σ) -modules, as in Definition 11.4, is closed under extensions and kernels. Furthermore, it is idempotent complete; concretely, any direct summand of an object in \mathcal{E} lies in \mathcal{E} .*

Proof. Suppose we are given a short exact sequence of R -modules

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

The long exact sequence for Tor tells us that if two of the terms lie in \mathcal{E} , then for all $i \in \mathbb{Z}$

$$\text{Tor}_i^R(\sigma^{-1}R, T') = \text{Tor}_i^R(\sigma^{-1}R, T) = \text{Tor}_i^R(\sigma^{-1}R, T'') = 0.$$

It is clear that if T'' and T are finitely presented R -modules of projective dimension ≤ 1 then so is T' , and that if T'' and T' are finitely presented R -modules of projective dimension ≤ 1 then so is T . Hence \mathcal{E} is closed under kernels and extensions.

Suppose now that T is an object of \mathcal{E} , and that as R -modules, $T = A \oplus B$. Since

$$0 = \text{Tor}_i^R(\sigma^{-1}R, T) = \text{Tor}_i^R(\sigma^{-1}R, A) \oplus \text{Tor}_i^R(\sigma^{-1}R, B),$$

we deduce that, for all $i \in \mathbb{Z}$, $\text{Tor}_i^R(\sigma^{-1}R, A) = \text{Tor}_i^R(\sigma^{-1}R, B) = 0$. The projective dimensions of A and B are bounded above by the projective dimension of T , which is ≤ 1 . Furthermore, we have an exact sequence

$$T \xrightarrow{e} T \longrightarrow A \longrightarrow 0$$

where e is an idempotent map on T . This expresses A as a quotient of two finitely presented modules. Hence A is finitely presented. Thus A lies in \mathcal{E} . □

Definition 11.7. The bounded derived category of the exact category \mathcal{E} , denoted $D^b(\mathcal{E})$, is defined as follows. The objects are bounded chain complexes of objects of \mathcal{E} . The morphisms are obtained from the chain maps by formally inverting that maps whose mapping cones are acyclic (as complexes of R -modules). There is an obvious functor $i : D^b(\mathcal{E}) \longrightarrow D(R)$.

□

Lemma 11.8. *The functor $i : D^b(\mathcal{E}) \longrightarrow D(R)$ is fully faithful.*

Proof. Let us begin by showing that, for any objects $T, T' \in \mathcal{E}$ and any $n \in \mathbb{Z}$,

$$\{D^b(\mathcal{E})\}(T, \Sigma^n T') = \{D(R)\}(T, \Sigma^n T') .$$

Take a map in $D^b(\mathcal{E})$ of the form $T \longrightarrow \Sigma^n T'$. There exists a bounded complex in \mathcal{E} which we call X , a quasi-isomorphism $g : X \longrightarrow T$, and a map of complexes $f : X \longrightarrow \Sigma^n T'$ so that our map is fg^{-1} . That is, we have a complex

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \xrightarrow{\partial_0} X^1 \longrightarrow \cdots$$

There is a quasi-isomorphism $X \longrightarrow T$; in particular $H^0(X) = T$. We have an exact sequence

$$\longrightarrow X^{-1} \longrightarrow \ker(\partial_0) \longrightarrow T \longrightarrow 0 .$$

But $T \in \mathcal{E}$ means that T is of projective dimension ≤ 1 . There is an exact sequence

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

with P and Q f.g. projective. Since P and Q are projective R -modules, there exists a map

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \longrightarrow & T & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow 1 & & \\ X^{-1} & \longrightarrow & \ker(\partial_0) & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

Let Z be given by the pushout square

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ X^{-1} & \longrightarrow & Z. \end{array}$$

The short exact sequence

$$0 \longrightarrow X^{-1} \longrightarrow Z \longrightarrow T \longrightarrow 0$$

establishes that $Z \in \mathcal{E}$ and that the complex $X^{-1} \longrightarrow Z$ is quasi-isomorphic to T . We deduce a quasi-isomorphism $h : X' \longrightarrow X$ of complexes, given below:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X^{-1} & \longrightarrow & Z & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots \end{array}$$

It follows that the map $fg^{-1} : T \longrightarrow \Sigma^n T'$ is equal to the map $\{fh\}\{gh\}^{-1}$. Since X' is concentrated in degrees 0 and 1, it follows that fh vanishes unless $n = 0$ or 1. Unless $n = 0$ or 1, we have proved that $\{D^b(\mathcal{E})\}(T, \Sigma^n T')$ vanishes. As for

$$\{D(R)\}(T, \Sigma^n T') = \text{Ext}_R^n(T, T') ,$$

it must vanish since the projective dimension of T is ≤ 1 . In other words, for $n \neq 0, 1$ the equality

$$\{D^b(\mathcal{E})\}(T, \Sigma^n T') = \{D(R)\}(T, \Sigma^n T')$$

is just because both sides vanish.

We leave to the reader to check that the two sides are equal also when $n = 0$ or 1 . For $n = 0$ both sides identify as $\mathcal{E}(T, T')$, while for $n = 1$ both sides identify as $\text{Ext}_R^1(T, T')$.

Let T be an object of \mathcal{E} . Consider next the full subcategory $\mathcal{T} \subset D^b(\mathcal{E})$ defined by

$$\text{Ob}(\mathcal{T}) = \left\{ Y \in \text{Ob}(D^b(\mathcal{E})) \mid \begin{array}{l} \forall n \in \mathbb{Z}, \\ \{D^b(\mathcal{E})\}(T, \Sigma^n Y) \longrightarrow \{D(R)\}(T, \Sigma^n Y) \\ \text{is an isomorphism} \end{array} \right\}.$$

By the above, \mathcal{T} contains \mathcal{E} , and clearly it is triangulated. Hence \mathcal{T} contains all of $D^b(\mathcal{E})$. Next, take any Y in $D^b(\mathcal{E})$, and consider the full subcategory $\mathcal{R} \subset D^b(\mathcal{E})$ given by

$$\text{Ob}(\mathcal{R}) = \left\{ X \in \text{Ob}(D^b(\mathcal{E})) \mid \begin{array}{l} \forall n \in \mathbb{Z}, \\ \{D^b(\mathcal{E})\}(X, \Sigma^n Y) \longrightarrow \{D(R)\}(X, \Sigma^n Y) \\ \text{is an isomorphism} \end{array} \right\}.$$

By the above, $\mathcal{E} \subset \mathcal{R}$, and \mathcal{R} is clearly triangulated. Hence \mathcal{R} contains $D^b(\mathcal{E})$. \square

Lemma 11.9. *Assume that maps in σ are all injections. The natural map $D^b(\mathcal{E}) \longrightarrow D(R)$ factors through $\mathcal{A}^c = D^c(R, \sigma) \subset D(R)$, and the induced map $D^b(\mathcal{E}) \longrightarrow \mathcal{A}^c$ is an equivalence of categories.*

Proof. Every object of \mathcal{E} is quasi-isomorphic to a complex $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$ of f.g. projectives; that is, $\mathcal{E} \subset \mathcal{B}^c$. Furthermore, every object $e \in \mathcal{E}$ satisfies $\{\sigma^{-1}R\}^L \otimes_R e = 0$, and by Proposition 5.3 this means $e \in \mathcal{A}$. Therefore $e \in \mathcal{A} \cap \mathcal{B}^c = \mathcal{A}^c$.

By Lemma 11.8, $D^b(\mathcal{E})$ is a full, triangulated subcategory of $\mathcal{B} = D(R)$. Clearly, it is the smallest full, triangulated subcategory containing \mathcal{E} . Since \mathcal{A}^c contains \mathcal{E} , it follows that $D^b(\mathcal{E}) \subset \mathcal{A}^c$.

We also know that the maps in σ are injections. If $s : P \longrightarrow Q$ lies in σ , then its cokernel lies in \mathcal{E} , and hence $D^b(\mathcal{E}) \subset \mathcal{A}^c$ contains all $s \in \sigma$. By 3.9.3, \mathcal{A}^c is the smallest thick subcategory of \mathcal{B} containing σ . If we could prove that $D^b(\mathcal{E})$ is thick, it would follow that $\mathcal{A}^c \subset D^b(\mathcal{E})$; we are reduced to proving that $D^b(\mathcal{E})$ is thick. But Lemma 11.6 tells us that \mathcal{E} is idempotent complete, and Theorem 2.8 of Balmer and Schlichting's [1] allows us to deduce that $D^b(\mathcal{E})$ is idempotent complete, hence thick. \square

Theorem 11.10. *Suppose every morphism in σ is injective. Then the algebraic K -theory of the Waldhausen category $\mathcal{A}^c = D^c(R, \sigma)$ is isomorphic to the algebraic K -theory of the exact category $\mathcal{E} = H(R, \sigma)$*

$$K_*(R, \sigma) = K_{*-1}(D^c(R, \sigma)) = K_{*-1}(H(R, \sigma)) .$$

Proof. By Lemma 11.9, the natural map $D^b(\mathcal{E}) \longrightarrow D(R)$ induces a triangulated equivalence of $D^b(\mathcal{E})$ with \mathcal{A}^c . Hence the induced map in K -theory is an isomorphism. But Waldhausen's $K_i(D^b(\mathcal{E}))$ agree with Quillen's $K_i(\mathcal{E})$. \square

12. ALGEBRAIC L -THEORY

We now extend our results to the algebraic L -theory of rings with involution. We refer to Ranicki [24], [25] more detailed expositions of algebraic L -theory.

An *involution* on a ring R is an anti-automorphism

$$R \longrightarrow R ; r \mapsto \bar{r} .$$

The involution is used to regard a left R -module M as a right R -module by

$$M \times R \longrightarrow M ; (x, r) \mapsto \bar{r}x .$$

The *dual* of a (left) R -module M is the R -module

$$M^* = \text{Hom}_R(M, R) , R \times M^* \longrightarrow M^* ; (r, f) \mapsto (x \mapsto f(x)\bar{r}) .$$

The *dual* of an R -module morphism $s : P \longrightarrow Q$ is the R -module morphism

$$s^* : Q^* \longrightarrow P^* ; f \mapsto (x \mapsto f(s(x))) .$$

If M is f.g. projective then so is M^* , and

$$M \longrightarrow M^{**} ; x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism which is used to identify $M^{**} = M$.

Hypothesis 12.1. *In this section, we assume that*

- (i) R is a ring with involution,
- (ii) the duals of morphisms $s : P \longrightarrow Q$ in σ are morphisms $s^* : Q^* \longrightarrow P^*$ in σ ,
- (iii) $\epsilon \in R$ is a central unit such that $\bar{\epsilon} = \epsilon^{-1}$ (e.g. $\epsilon = \pm 1$).

The noncommutative localization $\sigma^{-1}R$ is then also a ring with involution, with $\epsilon \in \sigma^{-1}R$ a central unit such that $\bar{\epsilon} = \epsilon^{-1}$. \square

We review briefly the chain complex construction of the f.g. projective ϵ -quadratic L -groups $L_*(R, \epsilon)$ and the ϵ -symmetric L -groups $L^*(R, \epsilon)$. Given an R -module chain complex C let the generator $T \in \mathbb{Z}_2$ act on the \mathbb{Z} -module chain complex $C \otimes_R C$ by the ϵ -transposition duality

$$T_\epsilon : C_p \otimes_R C_q \longrightarrow C_q \otimes_R C_p : x \otimes y \mapsto (-1)^{pq} \epsilon y \otimes x .$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z}

$$W : \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] .$$

The ϵ -symmetric (resp. ϵ -quadratic) Q -groups of C are the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of $C \otimes_R C$

$$\begin{aligned} Q^n(C, \epsilon) &= H^n(\mathbb{Z}_2; C \otimes_R C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)) , \\ Q_n(C, \epsilon) &= H_n(\mathbb{Z}_2; C \otimes_R C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_R C)) . \end{aligned}$$

The Q -groups are chain homotopy invariants of C . There are defined forgetful maps

$$\begin{aligned} 1 + T_\epsilon : Q_n(C, \epsilon) &\longrightarrow Q^n(C, \epsilon) ; \psi \mapsto (1 + T_\epsilon)\psi , \\ Q^n(C, \epsilon) &\longrightarrow H_n(C \otimes_R C) ; \phi \mapsto \phi_0 . \end{aligned}$$

For f.g. projective C the function

$$C \otimes_R C \longrightarrow \mathrm{Hom}_R(C^*, C) ; x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

is an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes, with $T \in \mathbb{Z}_2$ acting on $\mathrm{Hom}_R(C^*, C)$ by $\theta \mapsto \epsilon\theta^*$. The element $\phi_0 \in H_n(C \otimes_R C) = H_n(\mathrm{Hom}_R(C^*, C))$ is a chain homotopy class of R -module chain maps $\phi_0 : C^{n-*} \longrightarrow C$.

An n -dimensional ϵ -symmetric complex over R (C, ϕ) is a bounded f.g. projective R -module chain complex C together with an element $\phi \in Q^n(C, \epsilon)$. The complex (C, ϕ) is *Poincaré* if the R -module chain map $\phi_0 : C^{n-*} \longrightarrow C$ is a chain equivalence.

Example 12.2. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over R is essentially the same as a nonsingular ϵ -symmetric form (M, λ) over (R, σ) , with $M = (C_0)^*$ a f.g. projective R -module and

$$\lambda = \phi_0 : M \times M \longrightarrow R$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^* ; x \mapsto (y \mapsto \lambda(x, y))$$

is an R -module isomorphism.

□

See pp. 210–211 of [25] for the notion of an ϵ -symmetric (*Poincaré*) pair. The *boundary* of an n -dimensional ϵ -symmetric complex (C, ϕ) is the $(n-1)$ -dimensional ϵ -symmetric Poincaré complex

$$\partial(C, \phi) = (\partial C, \partial \phi)$$

with $\partial C = C(\phi_0 : C^{n-*} \longrightarrow C)_{*+1}$ and $\partial \phi$ as defined on p. 218 of [25]. The n -dimensional ϵ -symmetric L -group $L^n(R, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over R with C n -dimensional. In particular, $L^0(R, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric forms over R .

An n -dimensional ϵ -symmetric complex (C, ϕ) over R is $\sigma^{-1}R$ -Poincaré if the $\sigma^{-1}R$ -module chain map $\sigma^{-1}\phi_0 : \sigma^{-1}C^{n-*} \longrightarrow \sigma^{-1}C$ is a chain equivalence, in which case $\sigma^{-1}(C, \phi)$ is an n -dimensional ϵ -symmetric Poincaré complex over $\sigma^{-1}R$.

The n -dimensional ϵ -symmetric Γ -group $\Gamma^n(R \longrightarrow \sigma^{-1}R, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}R$ -Poincaré complexes (C, ϕ) over R such that $\sigma^{-1}C$ is chain equivalent to an n -dimensional induced f.g. projective $\sigma^{-1}R$ -module chain complex. The n -dimensional ϵ -symmetric L -group $L^n(R, \sigma, \epsilon)$ is the cobordism group of $(n-1)$ -dimensional ϵ -symmetric Poincaré complexes over R (C, ϕ) such that C is $\sigma^{-1}R$ -contractible, i.e. $\sigma^{-1}C \simeq 0$.

Similarly in the ϵ -quadratic case, with groups $L_n(R, \epsilon)$, $\Gamma_n(R \longrightarrow \sigma^{-1}R, \epsilon)$, $L_n(R, \sigma, \epsilon)$. The ϵ -quadratic L - and Γ -groups are 4-periodic

$$\begin{aligned} L_n(R, \epsilon) &= L_{n+2}(R, -\epsilon) = L_{n+4}(R, \epsilon) , \\ \Gamma_n(R \longrightarrow \sigma^{-1}R, \epsilon) &= \Gamma_{n+2}(R \longrightarrow \sigma^{-1}R, -\epsilon) = \Gamma_{n+4}(R \longrightarrow \sigma^{-1}R, \epsilon) , \\ L_n(R, \sigma, \epsilon) &= L_{n+2}(R, \sigma, -\epsilon) = L_{n+4}(R, \sigma, \epsilon) . \end{aligned}$$

Proposition 12.3. *For any ring with involution R and noncommutative localization $\sigma^{-1}R$ there is defined a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(R, \epsilon) \longrightarrow \Gamma^n(R \longrightarrow \sigma^{-1}R, \epsilon) \xrightarrow{\partial} L^n(R, \sigma, \epsilon) \longrightarrow L^{n-1}(R, \epsilon) \longrightarrow \cdots$$

Similarly in the ϵ -quadratic case, with an exact sequence

$$\cdots \longrightarrow L_n(R, \epsilon) \longrightarrow \Gamma_n(R \longrightarrow \sigma^{-1}R, \epsilon) \xrightarrow{\partial} L_n(R, \sigma, \epsilon) \longrightarrow L_{n-1}(R, \epsilon) \longrightarrow \cdots$$

Proof. The relative group of $L^n(R, \epsilon) \longrightarrow \Gamma^n(R \longrightarrow \sigma^{-1}R, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}R$ -Poincaré pairs over R $(f : C \longrightarrow D, (\delta\phi, \phi))$ with (C, ϕ) Poincaré. The effect of algebraic surgery on (C, ϕ) using this pair is a cobordant $(n-1)$ -dimensional ϵ -symmetric Poincaré complex (C', ϕ') with C' $\sigma^{-1}R$ -contractible. The function $(f : C \longrightarrow D, (\delta\phi, \phi)) \mapsto (C', \phi')$ defines an isomorphism between the relative group and $L^n(R, \sigma, \epsilon)$. \square

Define

$$I = \text{im}(K_0(R) \longrightarrow K_0(\sigma^{-1}R)) ,$$

the subgroup of $K_0(\sigma^{-1}R)$ consisting of the projective classes of the f.g. projective $\sigma^{-1}R$ -modules induced from f.g. projective R -modules. By definition, $L_I^n(\sigma^{-1}R, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes over $\sigma^{-1}R$ (B, θ) such that $[B] \in I$. There are evident morphisms of Γ - and L -groups

$$\begin{aligned} \sigma^{-1}\Gamma^* &: \Gamma^n(R \longrightarrow \sigma^{-1}R, \epsilon) \longrightarrow L_I^n(\sigma^{-1}R, \epsilon) ; (C, \phi) \mapsto \sigma^{-1}(C, \phi) , \\ \sigma^{-1}\Gamma_* &: \Gamma_n(R \longrightarrow \sigma^{-1}R, \epsilon) \longrightarrow L_n^I(\sigma^{-1}R, \epsilon) ; (C, \psi) \mapsto \sigma^{-1}(C, \psi) . \end{aligned}$$

In general, the morphisms $\sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ need not be isomorphisms, since a bounded f.g. projective $\sigma^{-1}R$ -module chain complex D with $[D] \in I$ need not be chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective R -module chain complex C (cf. the chain complex lifting problem considered in sections 6 and 10).

It was proved in Chapter 3 of Ranicki [24] that if $R \longrightarrow \sigma^{-1}R$ is an injective Ore localization then the morphisms $\sigma^{-1}Q^*, \sigma^{-1}Q_*, \sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ are isomorphisms, so that there are defined localization exact sequences for both the ϵ -symmetric and the ϵ -quadratic L -groups

$$\begin{aligned} \cdots \longrightarrow L^n(R, \epsilon) \longrightarrow L_n^I(\sigma^{-1}R, \epsilon) \xrightarrow{\partial} L^n(R, \sigma, \epsilon) \longrightarrow L^{n-1}(R, \epsilon) \longrightarrow \cdots, \\ \cdots \longrightarrow L_n(R, \epsilon) \longrightarrow L_n^I(\sigma^{-1}R, \epsilon) \xrightarrow{\partial} L_n(R, \sigma, \epsilon) \longrightarrow L_{n-1}(R, \epsilon) \longrightarrow \cdots. \end{aligned}$$

Special cases of these sequences were obtained by Milnor-Husemoller, Karoubi, Pardon, Smith, Carlsson-Milgram.

For any bounded f.g. projective R -module chain complex C the natural R -module chain map

$$\varinjlim_{(B, \beta)} B = G\pi(C) \longrightarrow \sigma^{-1}C$$

induces morphisms

$$\begin{aligned} \sigma^{-1}Q^* : \varinjlim_{(B, \beta)} Q^n(B, \epsilon) &= Q^n(G\pi(C), \epsilon) \longrightarrow Q^n(\sigma^{-1}C, \epsilon), \\ \sigma^{-1}Q_* : \varinjlim_{(B, \beta)} Q_n(B, \epsilon) &= Q_n(G\pi(C), \epsilon) \longrightarrow Q_n(\sigma^{-1}C, \epsilon) \end{aligned}$$

with the direct limits taken over all the bounded f.g. projective R -module chain complexes B with a chain map $\beta : C \longrightarrow B$ such that $\sigma^{-1}\beta : \sigma^{-1}C \longrightarrow \sigma^{-1}B$ is a $\sigma^{-1}R$ -module chain equivalence. The natural projection $D \otimes_R D \longrightarrow D \otimes_{\sigma^{-1}R} D$ is an isomorphism for any bounded f.g. projective $\sigma^{-1}R$ -module chain complex D (since this is already the case for $D = \sigma^{-1}R$), so the Q -groups of $\sigma^{-1}C$ are the same whether $\sigma^{-1}C$ is regarded as an R -module or $\sigma^{-1}R$ -module chain complex.

Theorem 12.4. (Vogel [32], Theorem 8.4) *For any ring with involution R and noncommutative localization $\sigma^{-1}R$ the morphisms*

$$\sigma^{-1}\Gamma_* : \Gamma_n(R \longrightarrow \sigma^{-1}R, \epsilon) \longrightarrow L_n^I(\sigma^{-1}R, \epsilon) ; (C, \psi) \mapsto \sigma^{-1}(C, \psi)$$

are isomorphisms, and there is a localization exact sequence of ϵ -quadratic L -groups

$$\cdots \longrightarrow L_n(R, \epsilon) \longrightarrow L_n^I(\sigma^{-1}R, \epsilon) \xrightarrow{\partial} L_n(R, \sigma, \epsilon) \longrightarrow L_{n-1}(R, \epsilon) \longrightarrow \cdots$$

Proof. By algebraic surgery below the middle dimension it suffices to consider only the special cases $n = 0, 1$. In effect, it was proved in [32] that $\sigma^{-1}Q_*$ is an isomorphism for 0- and 1-dimensional C . \square

It was claimed in Proposition 25.4 of Ranicki [25] that $\sigma^{-1}\Gamma^*$ is also an isomorphism, assuming (incorrectly) that the chain complex lifting problem can always be solved. However, we do have :

Theorem 12.5. *If $\sigma^{-1}R$ is a noncommutative localization of a ring with involution R which is stably flat over R , there is a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(R, \epsilon) \longrightarrow L^n_I(\sigma^{-1}R, \epsilon) \xrightarrow{\partial} L^n(R, \sigma, \epsilon) \longrightarrow L^{n-1}(R, \epsilon) \longrightarrow \cdots$$

Proof. By Theorem 10.8 for any bounded f.g. projective R -module chain complex C the natural R -module chain map $G\pi(C) \longrightarrow \sigma^{-1}C$ induces isomorphisms in homology

$$H_*(G\pi(C)) \cong H_*(\sigma^{-1}C) .$$

Thus the natural $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map

$$G\pi(C) \otimes_R G\pi(C) \longrightarrow \sigma^{-1}C \otimes_R \sigma^{-1}C = \sigma^{-1}C \otimes_{\sigma^{-1}R} \sigma^{-1}C$$

induces isomorphisms of ϵ -symmetric Q -groups

$$\sigma^{-1}Q^* : \varinjlim_{(B, \beta)} Q^n(B, \epsilon) \longrightarrow Q^n(\sigma^{-1}C, \epsilon)$$

(and also isomorphisms $\sigma^{-1}Q_*$ of ϵ -quadratic Q -groups). By Proposition 10.12 every n -dimensional induced f.g. projective $\sigma^{-1}R$ -module chain complex D is chain equivalent to $\sigma^{-1}C$ for an n -dimensional f.g. projective R -module chain complex C , with

$$Q^n(D, \epsilon) = Q^n(\sigma^{-1}C, \epsilon) = \varinjlim_{(B, \beta)} Q^n(B, \epsilon) .$$

It follows that the morphisms of ϵ -symmetric Γ - and L -groups

$$\sigma^{-1}\Gamma^* : \Gamma^n(R \longrightarrow \sigma^{-1}R, \epsilon) \longrightarrow L^n_I(\sigma^{-1}R, \epsilon) ; (C, \phi) \mapsto \sigma^{-1}(C, \phi)$$

are also isomorphisms, and the localization exact sequence is given by Proposition 12.3. \square

Hypothesis 12.6. *For the remainder of this section, we assume Hypothesis 12.1 and also that $R \longrightarrow \sigma^{-1}R$ is an injection.* \square

As in Proposition 11.2 it follows that all the morphisms in σ are injections.

We shall now obtain the L -theoretic analogue of the algebraic K -theory identification $K_*(R, \sigma) = K_{*-1}(H(R, \sigma))$ obtained in section 11, with $H(R, \sigma)$ the exact category of (R, σ) -modules. We generalize the results of Ranicki [24] and Vogel [31] to prove that under Hypotheses 12.1, 12.6 the relative L -groups $L^*(R, \sigma, \epsilon)$, $L_*(R, \sigma, \epsilon)$ in the L -theory localization exact sequences are the L -groups of $H(R, \sigma)$ with respect to the following duality involution.

Define the *torsion dual* of an (R, σ) -module M to be the (R, σ) -module

$$M^\wedge = \text{Ext}_R^1(M, R) ,$$

using the involution on R to define the left R -module structure. If M has f.g. projective R -module resolution

$$0 \longrightarrow P_1 \xrightarrow{s} P_0 \longrightarrow M \longrightarrow 0$$

with $s \in \sigma$ the torsion dual M^\wedge has the dual f.g. projective R -module resolution

$$0 \longrightarrow P_0^* \xrightarrow{s^*} P_1^* \longrightarrow M^\wedge \longrightarrow 0$$

with $s^* \in \sigma$.

Proposition 12.7. *Let $M = \text{coker}(s : P_1 \longrightarrow P_0)$, $N = \text{coker}(t : Q_1 \longrightarrow Q_0)$ be (R, σ) -modules.*

(i) *The adjoint of the pairing*

$$M \times M^\wedge \longrightarrow \sigma^{-1}R/R ; (g \in P_0, f \in P_1^*) \mapsto fs^{-1}g$$

defines a natural R -module isomorphism

$$M^\wedge \longrightarrow \text{Hom}_R(M, \sigma^{-1}R/R) ; f \mapsto (g \mapsto fs^{-1}g) .$$

(ii) *The natural R -module morphism*

$$M \longrightarrow M^{\wedge\wedge} ; x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism.

(iii) *There are natural identifications*

$$\begin{aligned} M \otimes_R N &= \text{Tor}_0^R(M, N) = \text{Ext}_R^1(M^\wedge, N) = H_0(P \otimes_R Q) , \\ \text{Hom}_R(M^\wedge, N) &= \text{Tor}_1^R(M, N) = \text{Ext}_R^0(M^\wedge, N) = H_1(P \otimes_R Q) . \end{aligned}$$

The functions

$$\begin{aligned} M \otimes_R N &\longrightarrow N \otimes_R M ; x \otimes y \mapsto y \otimes x , \\ \text{Hom}_R(M^\wedge, N) &\longrightarrow \text{Hom}_R(N^\wedge, M) ; f \mapsto f^\wedge \end{aligned}$$

determine transposition isomorphisms

$$T : \text{Tor}_i^R(M, N) \longrightarrow \text{Tor}_i^R(N, M) \quad (i = 0, 1) .$$

(iv) *For any finite subset $V = \{v_1, v_2, \dots, v_k\} \subset M \otimes_R N$ there exists an exact sequence of (R, σ) -modules*

$$0 \longrightarrow N \longrightarrow L \longrightarrow \oplus_k M^\wedge \longrightarrow 0$$

such that $V \subset \ker(M \otimes_R N \longrightarrow M \otimes_R L)$.

Proof. (i) Apply the snake lemma to the morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(P_0, R) & \longrightarrow & \text{Hom}_R(P_0, \sigma^{-1}R) & \longrightarrow & \text{Hom}_R(P_0, \sigma^{-1}R/R) \longrightarrow 0 \\ & & \downarrow s^* & & \downarrow s_1^* & & \downarrow s_2^* \\ 0 & \longrightarrow & \text{Hom}_R(P_1, R) & \longrightarrow & \text{Hom}_R(P_1, \sigma^{-1}R) & \longrightarrow & \text{Hom}_R(P_1, \sigma^{-1}R/R) \longrightarrow 0 \end{array}$$

with s^* injective, s_1^* an isomorphism and s_2^* surjective, to verify that the R -module morphism

$$M^\wedge = \text{coker}(s^*) \longrightarrow \text{Hom}_R(M, \sigma^{-1}R/R) = \ker(s_2^*)$$

is an isomorphism.

(ii) Immediate from the identification

$$s^{**} = s : (P_0)^{**} = P_0 \longrightarrow (P_1)^{**} = P_1 .$$

(iii) Exercise for the reader.

(iv) Lift each $v_i \in M \otimes_R N$ to an element

$$v_i \in P_0 \otimes_R Q_0 = \text{Hom}_R(P_0^*, Q_0) \quad (1 \leq i \leq k) .$$

The R -module morphism defined by

$$u = \begin{pmatrix} s^* & 0 & 0 & \dots & 0 \\ 0 & s^* & 0 & \dots & 0 \\ 0 & 0 & s^* & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & t \end{pmatrix} : U_1 = (\oplus_k P_0^*) \oplus Q_1 \longrightarrow U_0 = (\oplus_k P_1^*) \oplus Q_0$$

is in σ , so that $L = \text{coker}(u)$ is an (R, σ) -module with a f.g. projective R -module resolution

$$0 \longrightarrow U_1 \xrightarrow{u} U_0 \longrightarrow L \longrightarrow 0 .$$

The short exact sequence of 1-dimensional f.g. projective R -module chain complexes

$$0 \longrightarrow Q \longrightarrow U \longrightarrow \oplus_k P^{1-*} \longrightarrow 0$$

is a resolution of a short exact sequence of (R, σ) -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \oplus_k M^\wedge \longrightarrow 0 .$$

The first morphism in the exact sequence

$$\text{Tor}_1^R(M, \oplus_k M^\wedge) \longrightarrow M \otimes_R N \longrightarrow M \otimes_R L \longrightarrow M \otimes_R (\oplus_k M^\wedge) \longrightarrow 0$$

sends $1_i \in \text{Tor}_1^R(M, \oplus_k M^\wedge) = \oplus_k \text{Hom}_R(M^\wedge, M^\wedge)$ to $v_i \in \ker(M \otimes_R N \longrightarrow M \otimes_R L)$. \square

Given an (R, σ) -module chain complex C define the ϵ -symmetric (resp. ϵ -quadratic) torsion Q -groups of C to be the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of the ϵ -transposition involution $T_\epsilon = \epsilon T$ on the \mathbb{Z} -module chain complex $\text{Tor}_1^R(C, C) = \text{Hom}_R(C^\wedge, C)$

$$Q_{\text{tor}}^n(C, \epsilon) = H^n(\mathbb{Z}_2; \text{Tor}_1^R(C, C)) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Tor}_1^R(C, C))) ,$$

$$Q_n^{\text{tor}}(C, \epsilon) = H_n(\mathbb{Z}_2; \text{Tor}_1^R(C, C)) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (\text{Tor}_1^R(C, C))) .$$

There are defined forgetful maps

$$1 + T_\epsilon : Q_n^{\text{tor}}(C, \epsilon) \longrightarrow Q_{\text{tor}}^n(C, \epsilon) ; \quad \psi \mapsto (1 + T_\epsilon)\psi ,$$

$$Q_{\text{tor}}^n(C, \epsilon) \longrightarrow H_n(\text{Tor}_1^R(C, C)) ; \quad \phi \mapsto \phi_0 .$$

The element $\phi_0 \in H_n(\text{Tor}_1^R(C, C))$ is a chain homotopy class of R -module chain maps $\phi_0 : C^{n-\wedge} \longrightarrow C$.

An n -dimensional ϵ -symmetric complex over (R, σ) (C, ϕ) is a bounded (R, σ) -module chain complex C together with an element $\phi \in Q_{\text{tor}}^n(C, \epsilon)$. The complex (C, ϕ) is *Poincaré* if the R -module chain maps $\phi_0 : C^{\wedge} \rightarrow C$ are chain equivalences.

Example 12.8. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over (R, σ) is essentially the same as a nonsingular ϵ -symmetric linking form (M, λ) over (R, σ) , with $M = (C_0)^{\wedge}$ an (R, σ) -module and

$$\lambda = \phi_0 : M \times M \rightarrow \sigma^{-1}R/R$$

a sesquilinear pairing such that the adjoint

$$M \rightarrow M^{\wedge} ; x \mapsto (y \mapsto \lambda(x, y))$$

is an R -module isomorphism.

□

The n -dimensional torsion ϵ -symmetric L -group $L_{\text{tor}}^n(R, \sigma, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over (R, σ) , with C n -dimensional. In particular, $L_{\text{tor}}^0(R, \sigma, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric linking forms over (R, σ) .

Similarly in the ϵ -quadratic case, with torsion L -groups $L_n^{\text{tor}}(R, \sigma, \epsilon)$. The ϵ -quadratic torsion L -groups are 4-periodic

$$L_n^{\text{tor}}(R, \sigma, \epsilon) = L_{n+2}^{\text{tor}}(R, \sigma, -\epsilon) = L_{n+4}^{\text{tor}}(R, \sigma, \epsilon) .$$

Theorem 12.9. *If $R \rightarrow \sigma^{-1}R$ is injective the relative L -groups in the localization exact sequences of Proposition 12.3*

$$\begin{aligned} \cdots \longrightarrow L^n(R, \epsilon) \longrightarrow \Gamma^n(R \rightarrow \sigma^{-1}R, \epsilon) \xrightarrow{\partial} L^n(R, \sigma, \epsilon) \longrightarrow L^{n-1}(R, \epsilon) \longrightarrow \cdots \\ \cdots \longrightarrow L_n(R, \epsilon) \longrightarrow \Gamma_n(R \rightarrow \sigma^{-1}R, \epsilon) \xrightarrow{\partial} L_n(R, \sigma, \epsilon) \longrightarrow L_{n-1}(R, \epsilon) \longrightarrow \cdots \end{aligned}$$

are the torsion L -groups

$$L^*(R, \sigma, \epsilon) = L_{\text{tor}}^*(R, \sigma, \epsilon) ,$$

$$L_*(R, \sigma, \epsilon) = L_*^{\text{tor}}(R, \sigma, \epsilon) .$$

Proof. For any bounded (R, σ) -module chain complex T there exists a bounded f.g. projective R -module chain complex C with a homology equivalence $C \rightarrow T$. Working as in [31] there is defined a distinguished triangle of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$\Sigma \text{Tor}_1^R(T, T) \rightarrow C \otimes_R C \rightarrow T \otimes_R T \rightarrow \Sigma^2 \text{Tor}_1^R(T, T)$$

with \mathbb{Z}_2 acting by the ϵ -transposition T_{ϵ} on the \mathbb{Z} -module chain complex $\text{Tor}_1^R(T, T)$ and by the $(-\epsilon)$ -transpositions $T_{-\epsilon}$ on $C \otimes_R C$ and $T \otimes_R T$, inducing long exact sequences

$$\begin{aligned} \cdots \longrightarrow Q_{\text{tor}}^n(T, \epsilon) \longrightarrow Q^{n+1}(C, -\epsilon) \longrightarrow Q^{n+1}(T, -\epsilon) \longrightarrow Q_{\text{tor}}^{n-1}(T, \epsilon) \longrightarrow \cdots \\ \cdots \longrightarrow Q_n^{\text{tor}}(T, \epsilon) \longrightarrow Q_{n+1}(C, -\epsilon) \longrightarrow Q_{n+1}(T, -\epsilon) \longrightarrow Q_{n-1}^{\text{tor}}(T, \epsilon) \longrightarrow \cdots \end{aligned}$$

Passing to the direct limits over all the bounded (R, σ) -module chain complexes U with a homology equivalence $\beta : T \longrightarrow U$ use Proposition 12.7 (iv) to obtain

$$\begin{aligned} \varinjlim_{(U, \beta)} Q^{n+1}(U, -\epsilon) &= 0, \\ \varinjlim_{(U, \beta)} Q_{n+1}(U, -\epsilon) &= 0 \end{aligned}$$

and hence

$$\begin{aligned} \varinjlim_{(U, \beta)} Q_{\text{tor}}^n(U, \epsilon) &= Q^{n+1}(C, -\epsilon), \\ \varinjlim_{(U, \beta)} Q_n^{\text{tor}}(U, \epsilon) &= Q_{n+1}(C, -\epsilon). \end{aligned}$$

□

Remark 12.10. The identification $L_*(R, \sigma, \epsilon) = L_*^{\text{tor}}(R, \sigma, \epsilon)$ for noncommutative $\sigma^{-1}R$ was first obtained by Vogel [31].

□

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